IAS Winter School & Workshop on Advanced Concepts in Wave Physics:
Topology and Parity-Time Symmetries
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Topology in the momentum space and emergent phenomena:

from electrons waves in solids

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TOKYO TECH Pursuing Excellence

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and Chern number
§ 3 Various Hall effects by Berry curvature
§ 4 topological magnonic crystals
§ 5 topological plasmonic crystals

§ 1 Berry curvature

Eigenvalue equation dependent on parameter $\vec{k}$

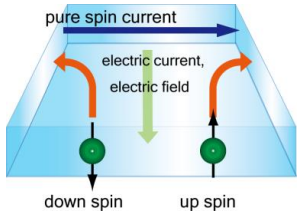
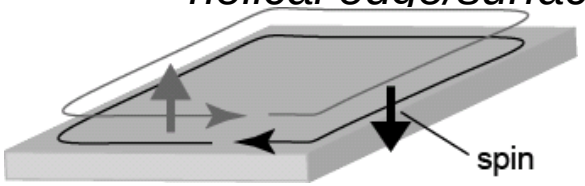
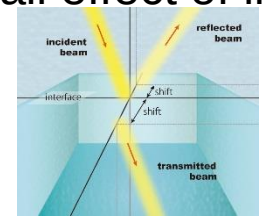
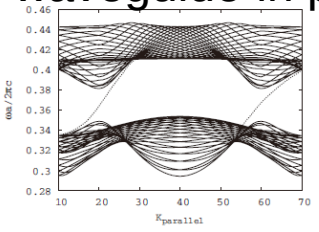
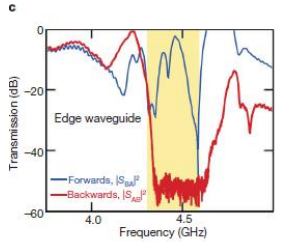
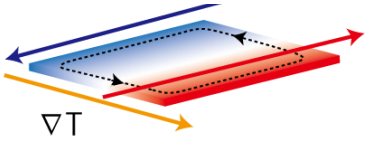
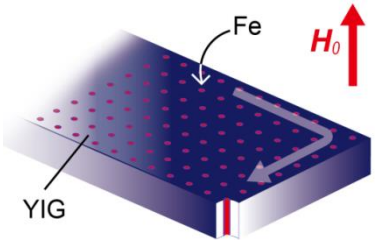
$$\hat{H}_{\vec{k}} u_{\vec{k}}(\vec{r}) = E_{\vec{k}} u_{\vec{k}}(\vec{r})$$



Berry curvature

$$B_{n,z}(\vec{k}) = i\left\langle \frac{\partial u_{n\vec{k}}}{\partial k_x} \middle| \frac{\partial u_{n\vec{k}}}{\partial k_y} \right\rangle + c.c$$

Phenomena due to Berry curvature of band structure

	Gapless — Hall effect	Gapped — topological phase edge/surface modes
Electrons	• Hall effect	→ Quantum Hall effect *chiral edge modes*
Electrons	• Spin Hall effect (of electrons) 	→ Topological insulators *helical edge/surface modes* 
Photons	•Spin Hall effect of light 	→ one-way waveguide in photonic crystal  
Magnons	• Magnon thermal Hall effect 	→ topological magnonic crystal 
Plasmons	• plasmon Hall effect ?	→ topological plasmonic crystal ?

Berry phase by an adiabatic change of a system

M.V.Berry, Proc. R. Soc. Lond. A392, 45 (1984)

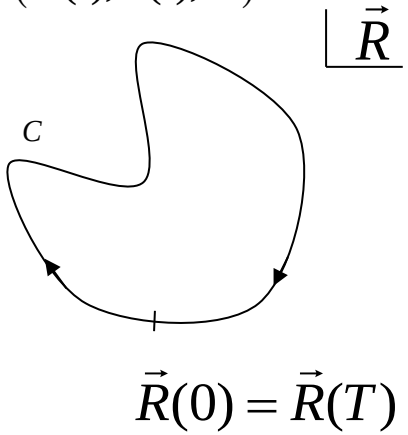
$H(\vec{R}) = H(X, Y, \cdots)$: dependent on time-dependent parameters $\vec{R}(t) = (X(t), Y(t), \cdots)$

$H(\vec{R})\left|\psi_n(\vec{R})\right\rangle = E_n(\vec{R})\left|\psi_n(\vec{R})\right\rangle$: n-th eigenstate

$t = 0$: $\vec{R}(0)$

↓ : Closed path

$t = T$: $\vec{R}(T) = \vec{R}(0)$



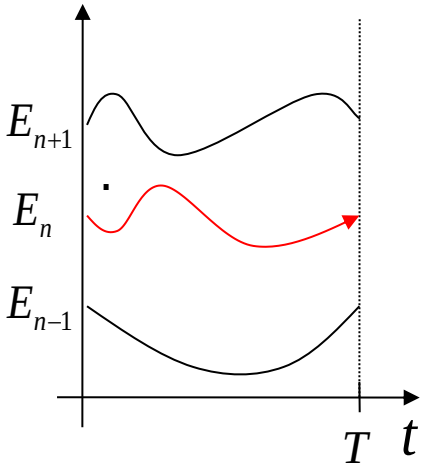
Initially, the state $\left|\psi(t=0)\right\rangle$ is one of the eigenstate $\left|\psi(t=0)\right\rangle = \left|\psi_n(\vec{R}(0))\right\rangle$.

Adiabatic change (T: very large) : the state remains the n-th eigenstate, provided there is no degeneracy

$$\left|\psi(t)\right\rangle \propto \left|\psi_n(\vec{R}(t))\right\rangle$$

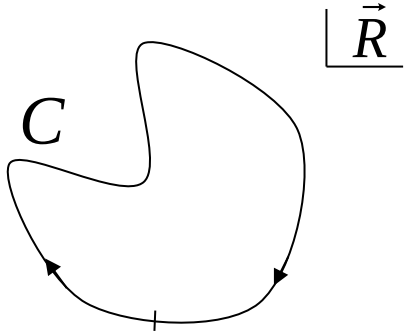
$$\left|\psi(t)\right\rangle = e^{i\gamma(t)} e^{\frac{i}{\hbar}\int_0^t E_n(\vec{R}(t))dt}\left|\psi_n(\vec{R}(t))\right\rangle$$

Berry phase : $\gamma(t) = i\int_0^t \left\langle\psi_n(\vec{R}(t))\right| \frac{d}{dt} \left|\psi_n(\vec{R}(t))\right\rangle dt$



Berry phase for the loop C (t=0→t=T)

$$\gamma(t) = i\int_0^t \left\langle \psi_n(\vec{R}(t)) \middle| \frac{d}{dt} \middle| \psi_n(\vec{R}(t)) \right\rangle dt$$



$$\gamma(C) = i\oint_C \left\langle \psi_n(\vec{R}(t)) \middle| \vec{\nabla}_R \middle| \psi_n(\vec{R}(t)) \right\rangle \cdot d\vec{R}$$: real

Properties:

1. It is determined by the shape of the loop C
   (does not depend on the time evolution $\vec{R}(t)$)

2. Gauge invariant
   Invariant under gauge transformation $\psi_n'(\vec{R}) = \psi_n(\vec{R})e^{i\theta_n(\vec{R})}$

For example, if $\vec{R} = (X, Y, Z)$

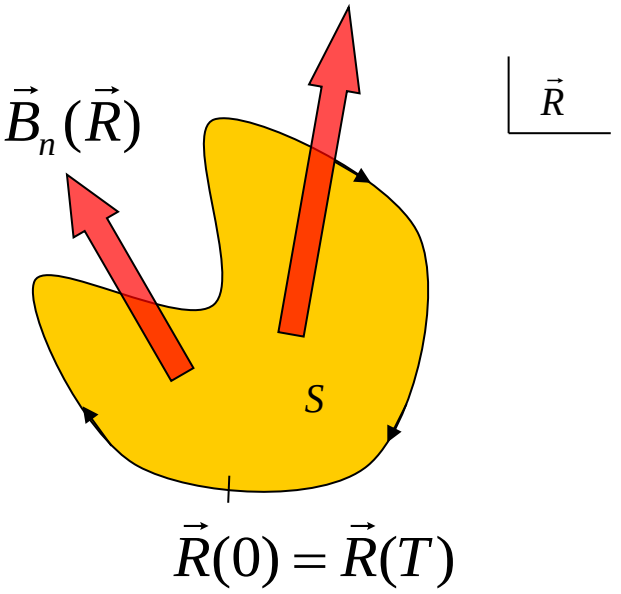
$$\gamma(C) = \oint_C \vec{A}_n \cdot d\vec{R} = \iint_S \vec{B}_n \cdot d\vec{S}$$

$\vec{A}_n(\vec{R}) = i\left\langle \psi_n(\vec{R}(t)) \middle| \vec{\nabla}_R \middle| \psi_n(\vec{R}(t)) \right\rangle$: Berry connection

$\vec{B}_n(\vec{R}) = \vec{\nabla} \times \vec{A}_n(\vec{R})$: Berry curvature

Rewritten as a surface integral

$$\begin{aligned}
(B_n)_i &= i\varepsilon_{ijk}\left\langle \partial_j \psi_n \middle| \partial_k \psi_n \right\rangle \\
&= i\varepsilon_{ijk} \sum_{m \neq n} \left\langle \partial_j \psi_n \middle| \psi_m \right\rangle \left\langle \psi_m \middle| \partial_k \psi_n \right\rangle \\
&= i\varepsilon_{ijk} \sum_{m \neq n} \frac{\left\langle \psi_n \middle| \partial_j H \middle| \psi_m \right\rangle \left\langle \psi_m \middle| \partial_k H \middle| \psi_n \right\rangle}{(E_n - E_m)^2}
\end{aligned}$$



where we used

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle \quad \Rightarrow \quad \partial_k H\left|\psi_n\right\rangle + H\left|\partial_k \psi_n\right\rangle = \partial_k E_n\left|\psi_n\right\rangle + E_n\left|\partial_k \psi_n\right\rangle$$

$$\Rightarrow \quad \left\langle \psi_m \middle| \partial_k \psi_n \right\rangle = \frac{\left\langle \psi_m \middle| \partial_k H \middle| \psi_n \right\rangle}{E_n - E_m}$$

Berry curvature in R space

Berry connection

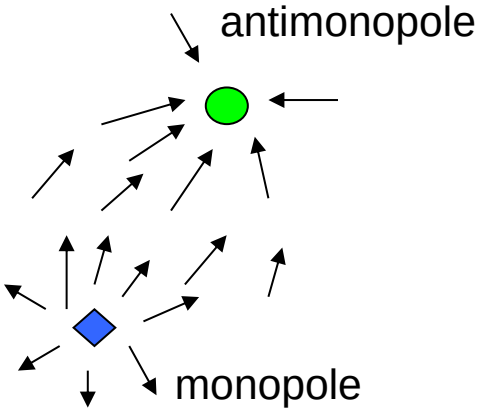
$$A_{ni}(\vec{R}) = i\left\langle n\vec{R}\left|\frac{\partial}{\partial R_i}\right|n\vec{R}\right\rangle$$

: "vector potential in R space"

Berry curvature

$$\vec{B}_n(\vec{R}) = \nabla_{\vec{R}} \times \vec{A}_n(\vec{R})$$

: "magnetic field in R-space"



Monopole density

$$\rho_n(\vec{R}) = \nabla_{\vec{k}} \cdot \vec{B}_n(\vec{R})$$

$$u_n'(\vec{R}) = u_n(\vec{R})e^{-i\chi(\vec{R})}$$: gauge transformation

$$\begin{cases} \vec{A}_n'(\vec{R}) = \vec{A}_n(\vec{R}) + \nabla_{\vec{R}}\chi \\ \vec{B}_n'(\vec{R}) = \vec{B}_n(\vec{R}) \\ \rho_n'(\vec{R}) = \rho_n(\vec{R}) \end{cases}$$

Similar to magnetic field !

Quantization of monopole charge

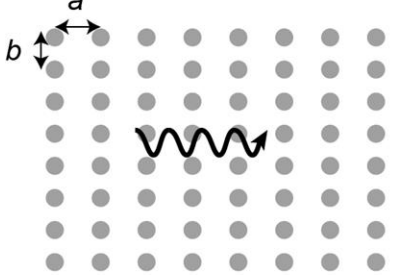
$$\rho_n(\vec{R}) = \sum_l q_l \delta(\vec{R} - \vec{R}_l)$$ q_l : integer

Waves in spatially periodic systems:

-- Berry curvature in k-space –

(e.g.) electrons in a crystal

Bloch theorem



Schrodinger eq. in crystals

$$\left[\frac{p^2}{2m}+V(\vec{r})\right]\psi(\vec{r})=E\psi(\vec{r})$$

$$V(\vec{r})=V(\vec{r}+\vec{a})$$

$\vec{a}$: **periodic potential**

Bloch theorem

Wavefunction labeled by wavevector $\vec{k}$

$$\psi_{\vec{k}}(\vec{r})=e^{i\vec{k}\cdot\vec{r}}u_{\vec{k}}(\vec{r})$$

$u_{\vec{k}}(\vec{r})$: periodic function

Eigenvalue eq. for $u_{\vec{k}}(\vec{r})$

$$\left[\frac{\left(-i\hbar\nabla+\hbar\vec{k}\right)^2}{2m}+V(\vec{r})\right]u_{\vec{k}}(\vec{r})=E_{\vec{k}}u_{\vec{k}}(\vec{r}) \longrightarrow \hat{H}_{\vec{k}}u_{\vec{k}}(\vec{r})=E_{\vec{k}}u_{\vec{k}}(\vec{r})$$

From the k-dependent eigenvalue eq. $\hat{H}_{\vec{k}}u_{\vec{k}}(\vec{r})=E_{\vec{k}}u_{\vec{k}}(\vec{r})$

one can define the Berry curvature $\vec{B}_n(\vec{k})=\mathrm{i}\left\langle\frac{\partial u_{n\vec{k}}}{\partial k_x}\middle|\frac{\partial u_{n\vec{k}}}{\partial k_y}\right\rangle+c.c.$

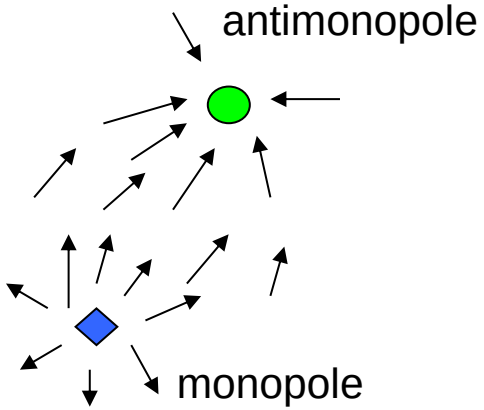
Berry curvature in k space

- **Berry curvature** $\vec{B}_n(\vec{k}) = \mathrm{i}\left\langle \frac{\partial u_{n\vec{k}}}{\partial k_x} \middle| \frac{\partial u_{n\vec{k}}}{\partial k_y} \right\rangle + c.c.$

$u_{n\vec{k}}$: periodic part of the Bloch wf.

$$\psi_{n\vec{k}}(\vec{x}) = u_{n\vec{k}}(\vec{x}) e^{i\vec{k}\cdot\vec{x}}$$ (n : band index)

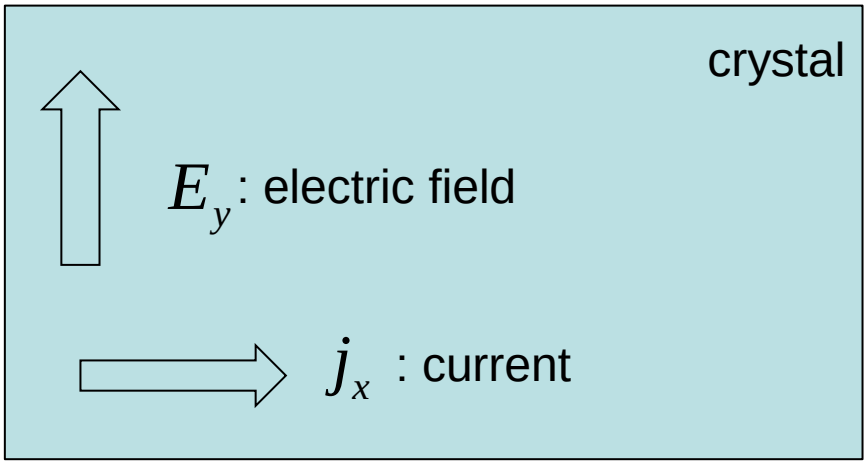
: "magnetic field in k-space"



- **Monopole density**

$$\rho_n(\vec{k}) = \nabla_{\vec{k}} \cdot \vec{B}_n(\vec{k})$$

§ 2
integer quantum Hall effect and Chern number



$$j_x = \sigma_{xy} E_y$$

σ_{xy} : Hall conductivity

→ Expressed in terms of Berry curvature $\vec{B}_n(\vec{k})$

Kubo formula for Hall conductivity (noninteracting electron, T=0)

Thouless et al. PRL49,405(1982), Kohmoto, Ann.Phys.(NY) 160,343 (1985)

2D system + electric field along y: $\vec{E} = (0, E)$

Perturbation theory $|\alpha'\rangle = |\alpha\rangle + \sum_{\beta(\neq\alpha)} \frac{\langle\beta|eEy|\alpha\rangle}{E_\alpha - E_\beta}|\beta\rangle$

→ Current density

$$\langle j_x \rangle = \frac{1}{L^2}\sum_\alpha f(E_\alpha)\langle\alpha|j_x|\alpha\rangle = \frac{1}{L^2}\sum_\alpha f(E_\alpha)\sum_{\beta(\neq\alpha)} \frac{\langle\alpha|(-ev_x)|\beta\rangle\langle\beta|eEy|\alpha\rangle}{E_\alpha - E_\beta} + c.c.$$

We then use $\langle\beta|v_y|\alpha\rangle = \langle\beta|\dot{y}|\alpha\rangle = \frac{-i}{\hbar}\langle\beta|[y,H]|\alpha\rangle = \frac{-i}{\hbar}(E_\alpha - E_\beta)\langle\beta|y|\alpha\rangle$

Hall conductivity

$$\sigma_{xy} = \frac{-ie^2\hbar}{L^2}\sum_\alpha f(E_\alpha)\sum_{\beta(\neq\alpha)} \frac{\langle\alpha|v_x|\beta\rangle\langle\beta|v_y|\alpha\rangle}{(E_\alpha - E_\beta)^2} + c.c.$$

$|\alpha\rangle = (n,\vec{k}), |\beta\rangle = (m,\vec{k})$ m,n :band index

$\vec{k}$: Bloch wavevector

Hall conductivity

$$\sigma_{xy} = \frac{-ie^2\hbar}{L^2}\sum_{\alpha} f(E_\alpha)\sum_{\beta(\neq\alpha)} \frac{\langle\alpha|v_x|\beta\rangle\langle\beta|v_y|\alpha\rangle}{\left(E_\alpha - E_\beta\right)^2} + c.c.$$

$|\alpha\rangle = (n,\vec{k}), |\beta\rangle = (m,\vec{k})$ m,n : band index

$\vec{k}$: Bloch wavevector

$$v_i = \frac{1}{\hbar}\frac{\partial H}{\partial k_i}$$

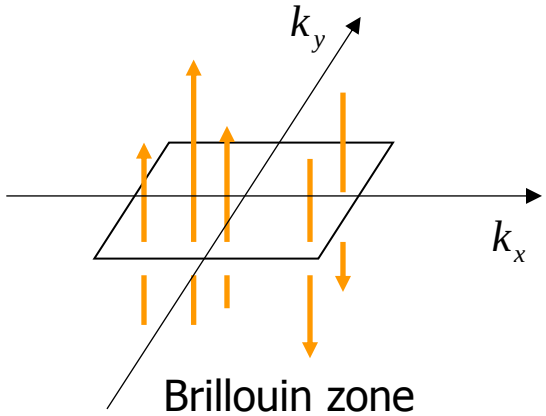
$$\sigma_{xy} = \frac{-ie^2}{\hbar L^2}\sum_{\vec{k},n} f(E_{n\vec{k}})\sum_{m(\neq n)} \frac{\left\langle n\vec{k}\left|\frac{\partial H}{\partial k_x}\right|m\vec{k}\right\rangle\left\langle m\vec{k}\left|\frac{\partial H}{\partial k_y}\right|n\vec{k}\right\rangle}{\left(E_{n\vec{k}} - E_{m\vec{k}}\right)^2} + c.c.$$

Hall conductivity

$$\sigma_{xy} = \frac{-e^2}{h}\int_{BZ}\frac{d^2k}{(2\pi)^2}\sum_n f(E_{n\vec{k}})B_{n,z}(\vec{k})$$

$$B_{n,z}(\vec{k}) = i\left\langle\frac{\partial u_{n\vec{k}}}{\partial k_x}\middle|\frac{\partial u_{n\vec{k}}}{\partial k_y}\right\rangle + c.c$$: Berry curvature

$f(E_{n\vec{k}})$: Fermi distribution

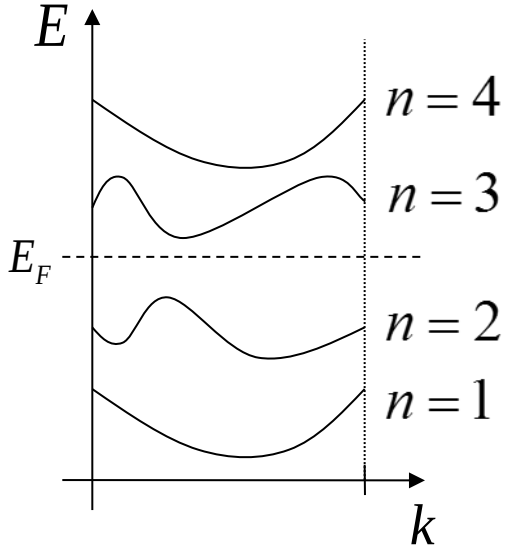


Quantization of Hall conductivity for insulators

Thouless et al. PRL49,405(1982), Kohmoto, Ann.Phys.(NY) 160,343 (1985)

$$\sigma_{xy} = \frac{-e^2}{h}\int_{BZ}\frac{d^2k}{(2\pi)^2}\sum_n f(E_{n\vec{k}})B_{n,z}(\vec{k})$$

$$B_{n,z}(\vec{k}) = i\left\langle \frac{\partial u_{n\vec{k}}}{\partial k_x}\middle|\frac{\partial u_{n\vec{k}}}{\partial k_y}\right\rangle + c.c$$



for gapped system (insulator) :

no states on the Fermi energy

$$\sigma_{xy} = \frac{-e^2}{h}\sum_{n\in\text{filled band}} Ch_n$$

$$Ch_{(n)} = \frac{1}{2\pi}\int_{BZ} B_{(n)z}(\vec{k})d^2k$$: Chern number of the n-th band : always an integer

Integer quantum Hall effect $\sigma_{xy} = \frac{e^2}{h}\cdot(\text{integer})$

Chern number is always an integer

$$Ch_n = \int_{BZ} \frac{d^2k}{2\pi} B_{n,z}(\vec{k}) \qquad \left(\vec{A}_n(\vec{k}) = i\left\langle u_{n\vec{k}} \middle| \frac{\partial}{\partial \vec{k}} \middle| u_{n\vec{k}} \right\rangle, \vec{B}_n(\vec{k}) = \nabla_{\vec{k}} \times \vec{A}_n(\vec{k}) \right)$$

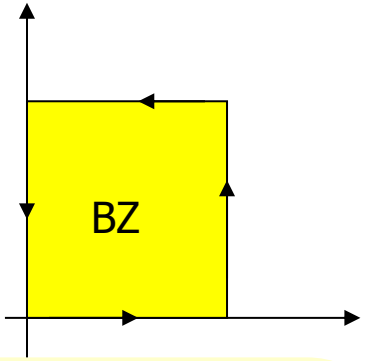
(Proof):

Phase of the Bloch wavefunction $u_{n\vec{k}}(x,y)$ is important

From Stokes theorem…

$$Ch_n = \int_{BZ} \frac{d^2k}{2\pi} B_{n,z}(\vec{k}) = \oint_{\partial BZ} \frac{d\vec{k}}{2\pi} \times \vec{A}_n(\vec{k}) = 0$$

BZ periodicity



This is not correct when $u_{n\vec{k}}(x,y)$ cannot be chosen as a single function over the BZ.

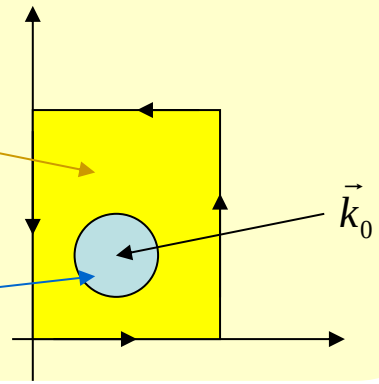
$u_{n\vec{k}}(\vec{k}) = e^{i\theta(\vec{k})} u'_{n\vec{k}}(\vec{k})$ Divide BZ into two patches

$$Ch_n = \frac{1}{2\pi} \oint_C \vec{A}_n(\vec{k}) \times d\vec{k} - \frac{1}{2\pi} \oint_C \vec{A}_n(\vec{k}) \times d\vec{k}$$

$$= \frac{1}{2\pi} \oint_C \frac{\partial q(\vec{k})}{\partial \vec{k}} \times d\vec{k} = \text{integer}$$

$u_{n\vec{k}}(\vec{k})$ $\vec{A}(\vec{k})$

$u'_{n\vec{k}}(\vec{k})$ $\vec{A}'(\vec{k})$

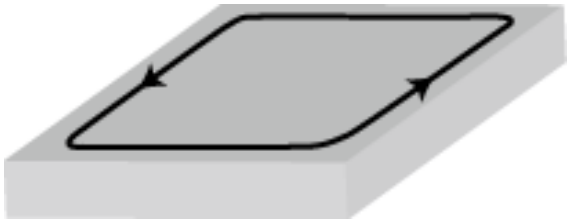


Chern number & topological chiral modes

Band gap → Chern number for n-th band = integer

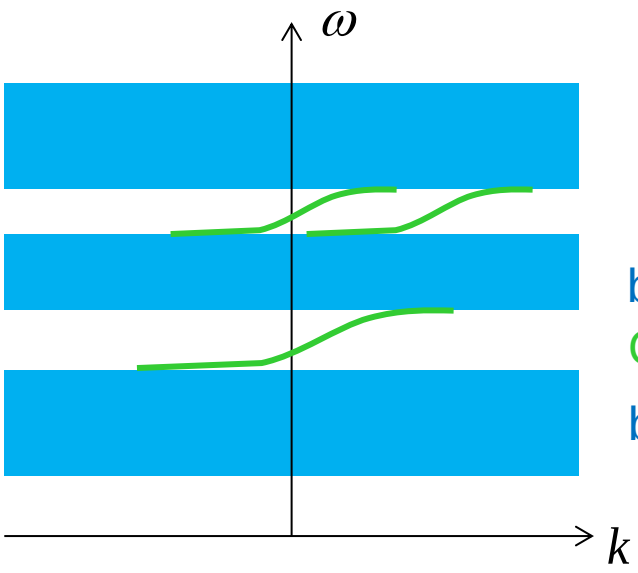
$$\mathrm{Ch}_n = \grave{o}_{BZ} \frac{d^2k}{2p} W_n\left(\overset{r}{k}\right)$$

Berry curvature



topological chiral edge modes

$$\sum_{n\hat{I}\ \text{bands below } E} \mathrm{Ch}_n = N^{o}\ \#(\text{clockwise chiral edge states in the gap at } E)$$



bulk mode: Chern number= Ch_3

(Ch_1+Ch_2) topological edge modes

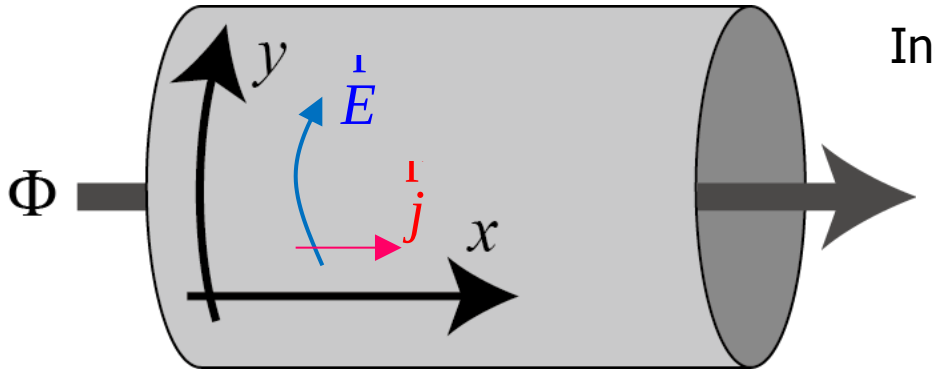
bulk mode: Chern number= Ch_2

Ch_1 topological edge modes

bulk mode: Chern number= Ch_1

Relation between the number of edge states and Chern number:

Laughlin gedanken experiment:



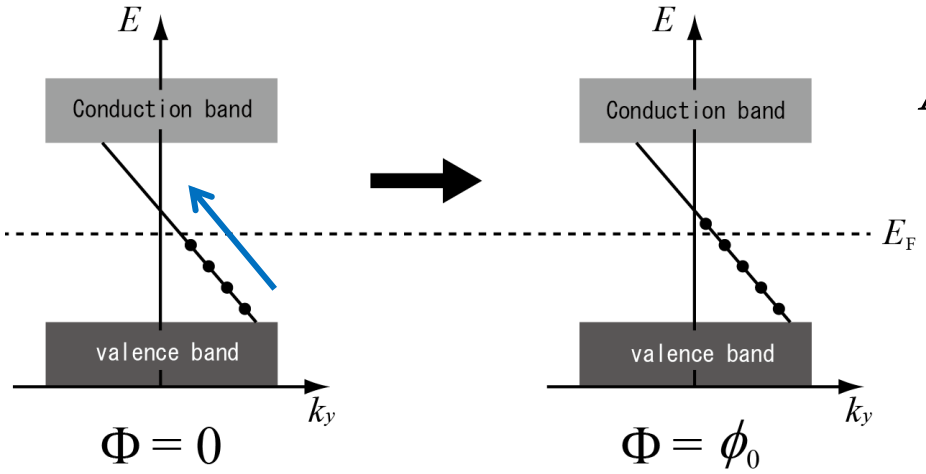
Increase the flux
→ charge transport along x

$$F = 0 \text{ at } t = 0$$

$$F = f_0 \text{ at } t = T$$

$$E_y = \frac{f_0}{TL_y} \quad \therefore \quad j_x = s_{xy}\frac{f_0}{TL_y} = \frac{-e^2}{h}\mathrm{Ch}\frac{f_0}{TL_y} = -\frac{e}{TL_y}\mathrm{Ch}$$

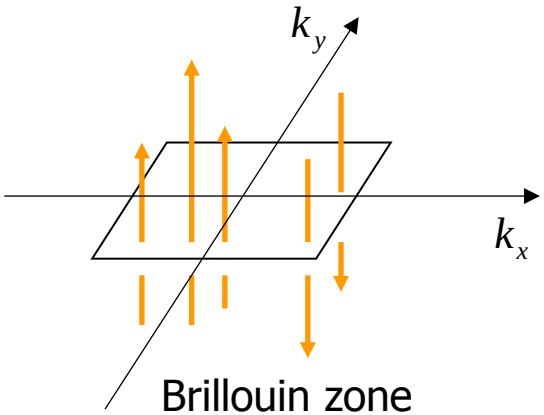
$$\therefore \quad Q = -e \times \mathrm{Ch}$$ Number of electron carried from left end to right end = Ch



$$A_y = -\frac{F}{L_y}$$ Gradual change of vector pot.
= change of wavenumber

This charge transport is between the edge states on the left and right ends.
→ gapless edge modes exist.

Edge states at the right end



Brillouin zone

$$\sigma_{xy} = \frac{-e^2}{h} \sum_{n \in \text{filled band}} Ch_n$$

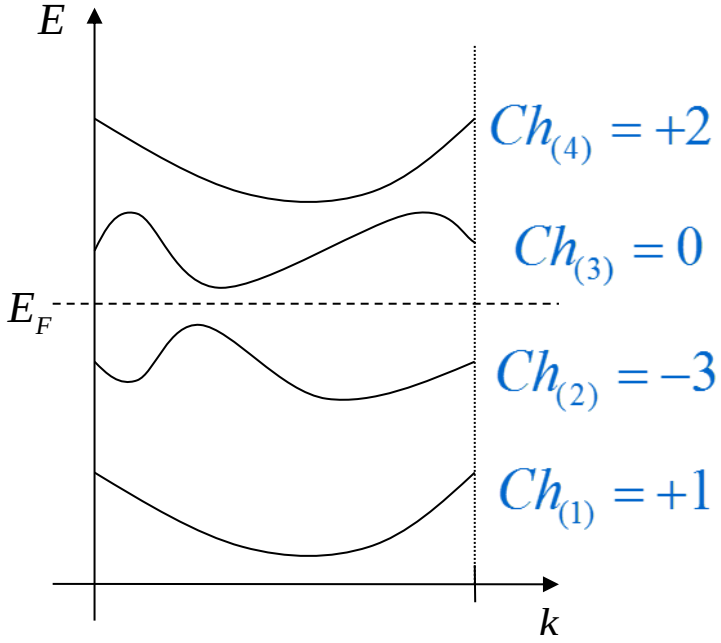
$$Ch_{(n)} = \frac{1}{2\pi} \iint_{BZ} B_{(n)z}(\vec{k}) d^2k$$

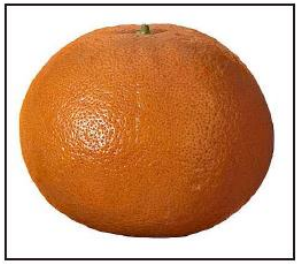
= Total "flux" inside the Brilouin zone

Topological number :

Invariant under continuous change of the system

(Example)





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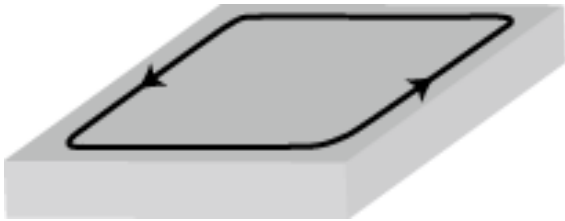
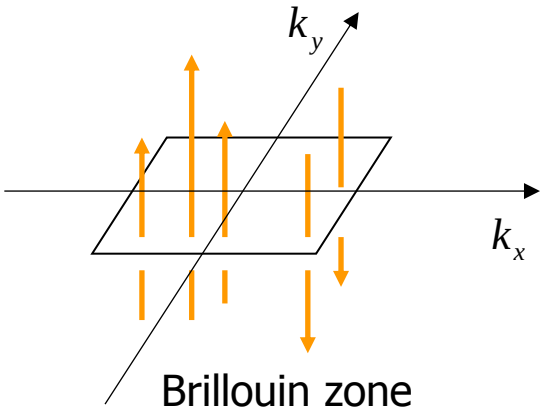
$$s_{xy} = \frac{e^2}{h} \sum_{n \in \text{filled band}} Ch_n$$

Number of chiral edge modes

$$\sigma_{xy} = \frac{-e^2}{h} \sum_{n \in \text{filled band}} Ch_n = \frac{-e^2}{h}(+1-3) = \frac{2e^2}{h}$$





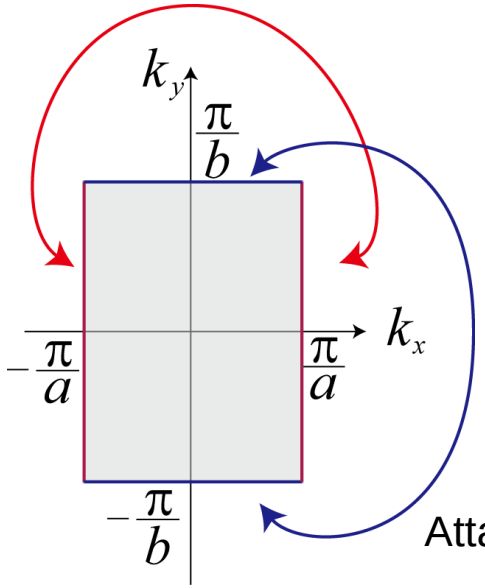
Brillouin zone

$$\sigma_{xy} = \frac{-e^2}{h} \sum_{n \in \text{filled band}} Ch_n$$

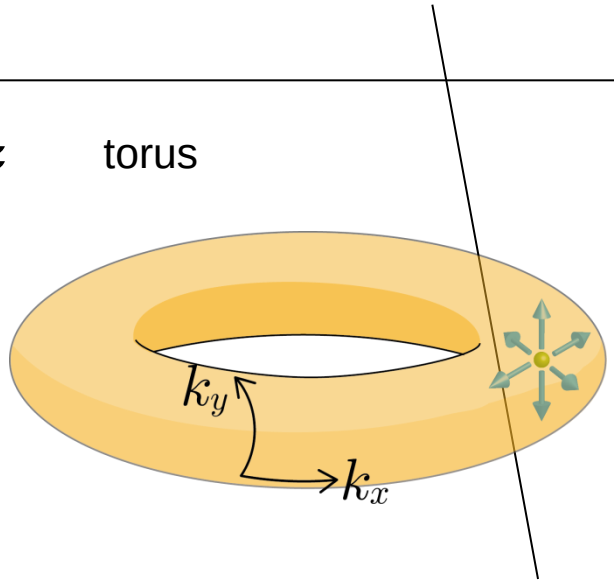
$$Ch_{(n)} = \frac{1}{2\pi} \iint_{BZ} B_{(n)z}(\vec{k}) d^2k$$

= Total "flux" inside the Brilouin zone

2D Brillouin zone $\approx$ torus



Attach

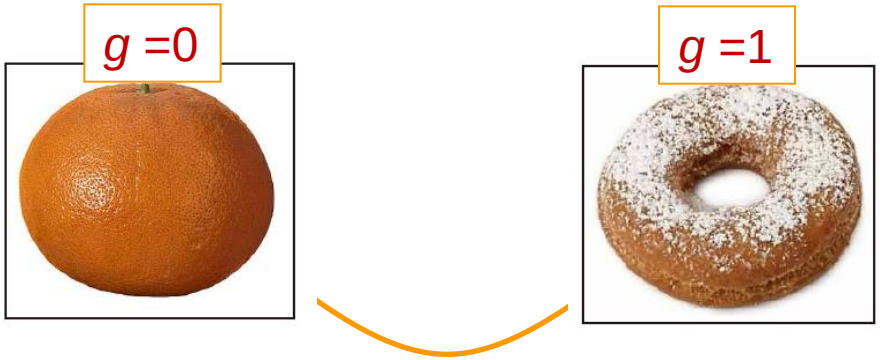


Total flux through torus surface
= number of magnetic monopoles inside the torus

Topological number :
Invariant under continuous change of the system

Quantization of Hall conductivity
＝quantization of monopole charge

Topology & topological number



They cannot be continuously deformed to each other

Classified by genus *g* (number of holes)
genus= topological number
(unchanged by continuous deformation)

$$2-2g=\frac{1}{2\pi}\int_S d^2r\,\Omega(\vec{r})$$

Gauss curvature

genus

A magnonic band (separated by a gap)

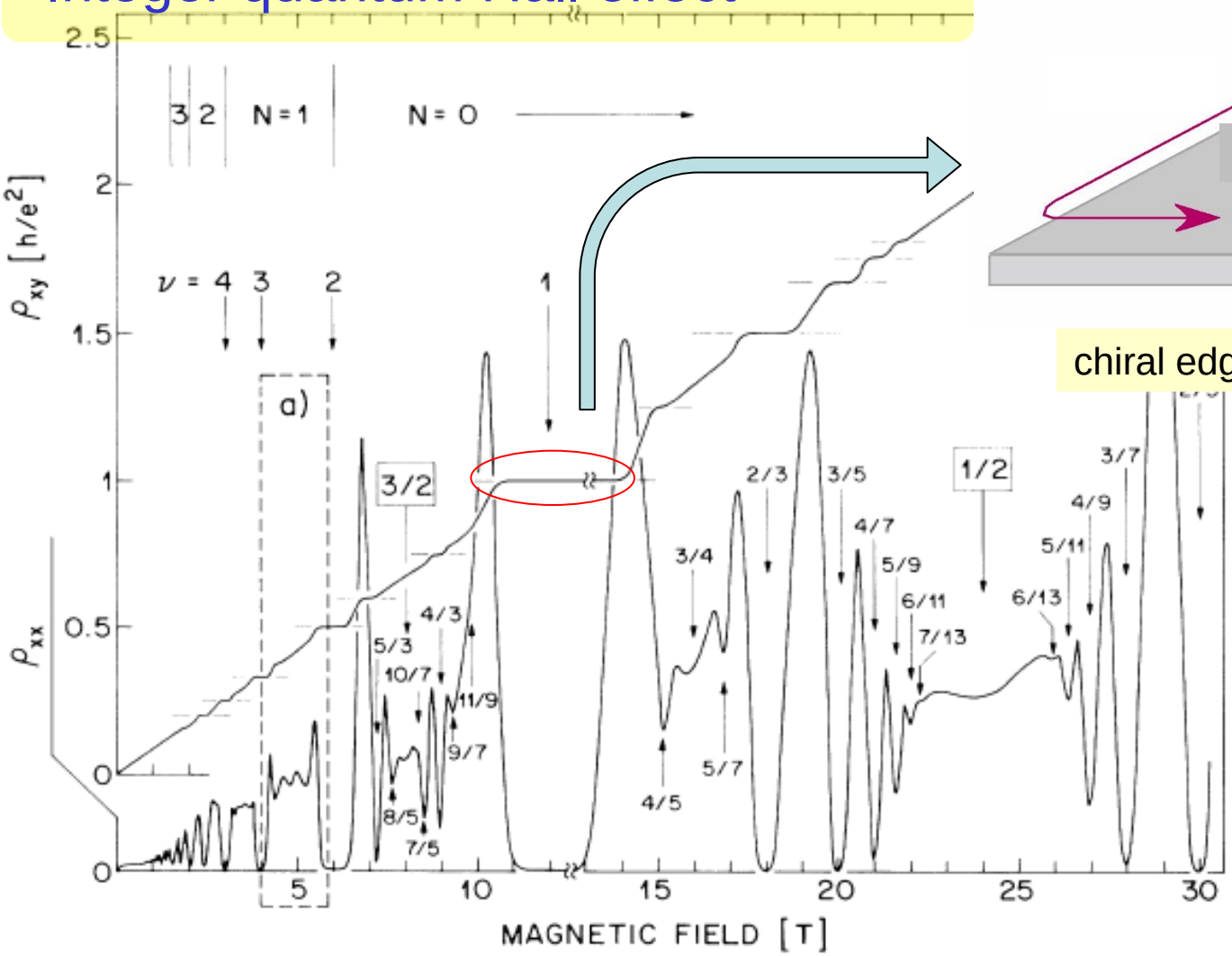
Classified by Chern number C_j
Chern number= topological number
(unchanged by continuous deformation)

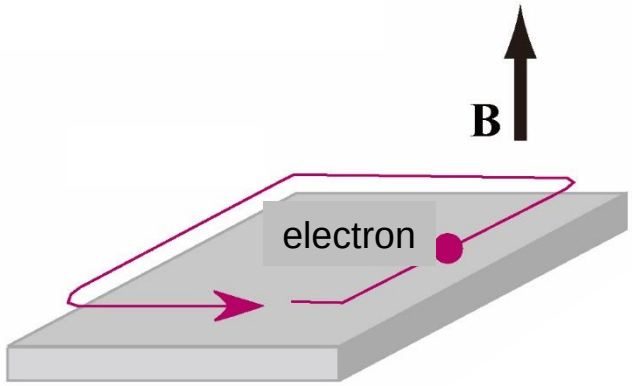
$$C_j=\frac{1}{2\pi}\int_{BZ} d^2k\,\Omega_j(\vec{k})$$

Berry curvature

Number of chiral edge modes

Integer quantum Hall effect





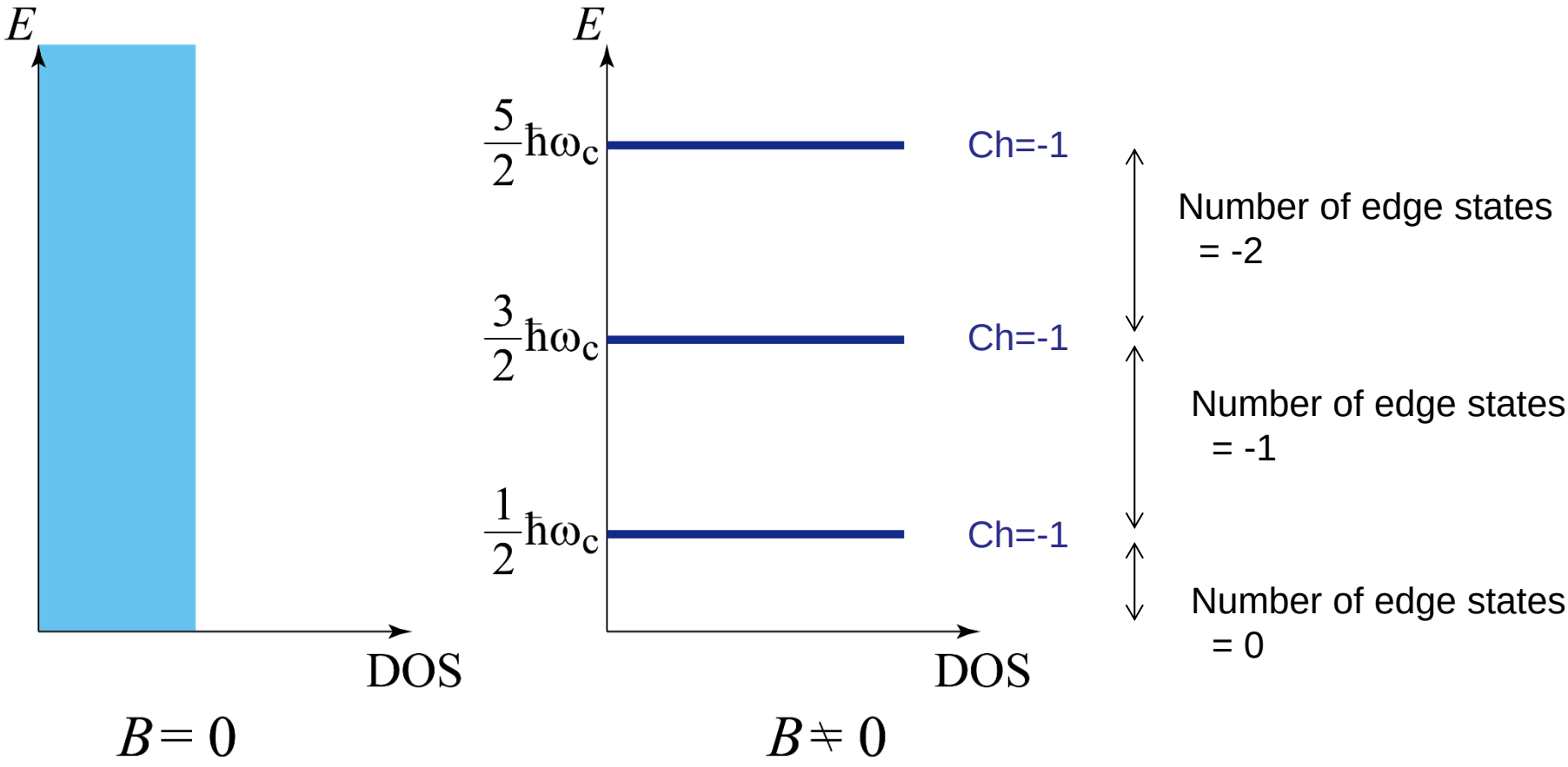
chiral edge state

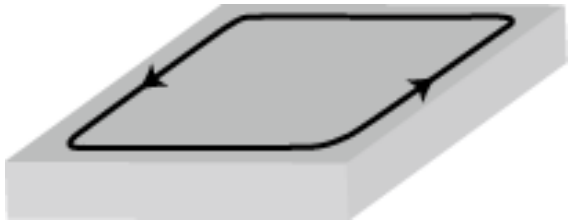
R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui
A. C. Gossard, J. H. English

Phys. Rev. Lett. 59, 1776 - 1779 (1987)

2D electron gas in a magnetic field

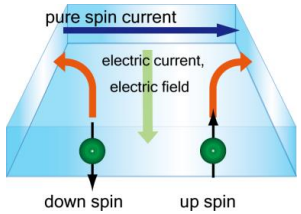
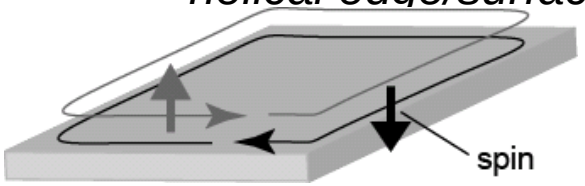
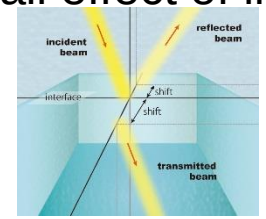
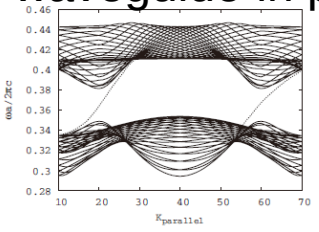
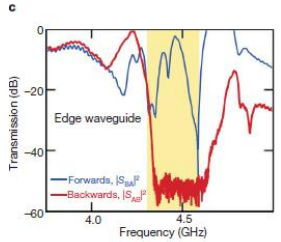
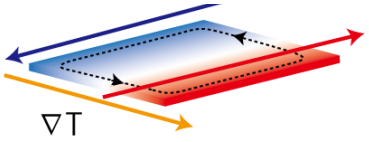
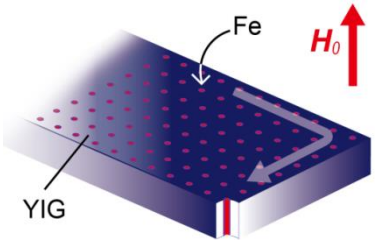
Electronic states form Landau levels with spacing $\hbar w_c$





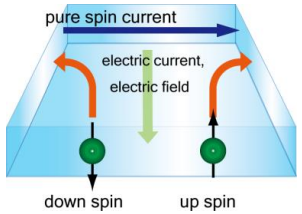
§ 3 Various Hall effects by Berry curvature

Phenomena due to Berry curvature of band structure

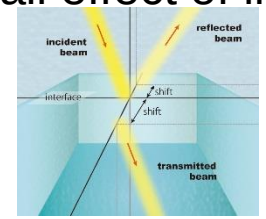
	Gapless — Hall effect	Gapped — topological phase edge/surface modes
Electrons	• Hall effect	→ Quantum Hall effect *chiral edge modes*
Electrons	• Spin Hall effect (of electrons) 	→ Topological insulators *helical edge/surface modes* 
Photons	•Spin Hall effect of light 	→ one-way waveguide in photonic crystal  
Magnons	• Magnon thermal Hall effect 	→ topological magnonic crystal 
Plasmons	• plasmon Hall effect ?	→ topological plasmonic crystal ?

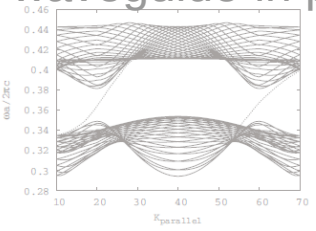
Phenomena due to Berry curvature of band structure

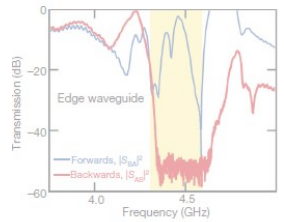
	Gapless	Hall effect	Gapped	topological phase edge/surface modes
Electrons	• Hall effect	→	Quantum Hall effect	*chiral edge modes*
Electrons	• Spin Hall effect (of electrons)	→	Topological insulators	*helical edge/surface modes*
Photons	•Spin Hall effect of light	→	one-way waveguide in photonic crystal	
Magnons	• Magnon thermal Hall effect	→	topological magnonic crystal	
Plasmons	• plasmon Hall effect ?	→	topological plasmonic crystal ?	

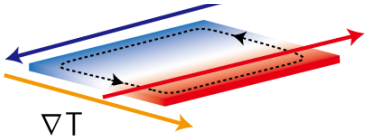


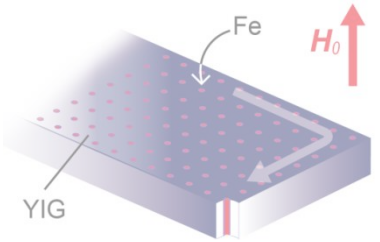












Semiclassical theory

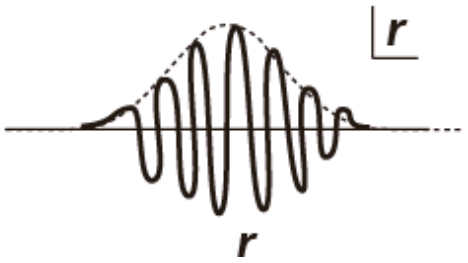
Adams, Blount; Sundaram,Niu, …

Motion of a wavepacket under slowly varying background

For electrons:

$$\begin{cases} \dot{\vec{x}} = \dfrac{1}{\hbar}\dfrac{\partial E_n(\vec{k})}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Omega}_n(\vec{k}) \\ \dot{\vec{k}} = -e\vec{E} \end{cases}$$

Boltzmann transport



$\dot{\vec{k}}$: force onto the electron

→ transverse velocity

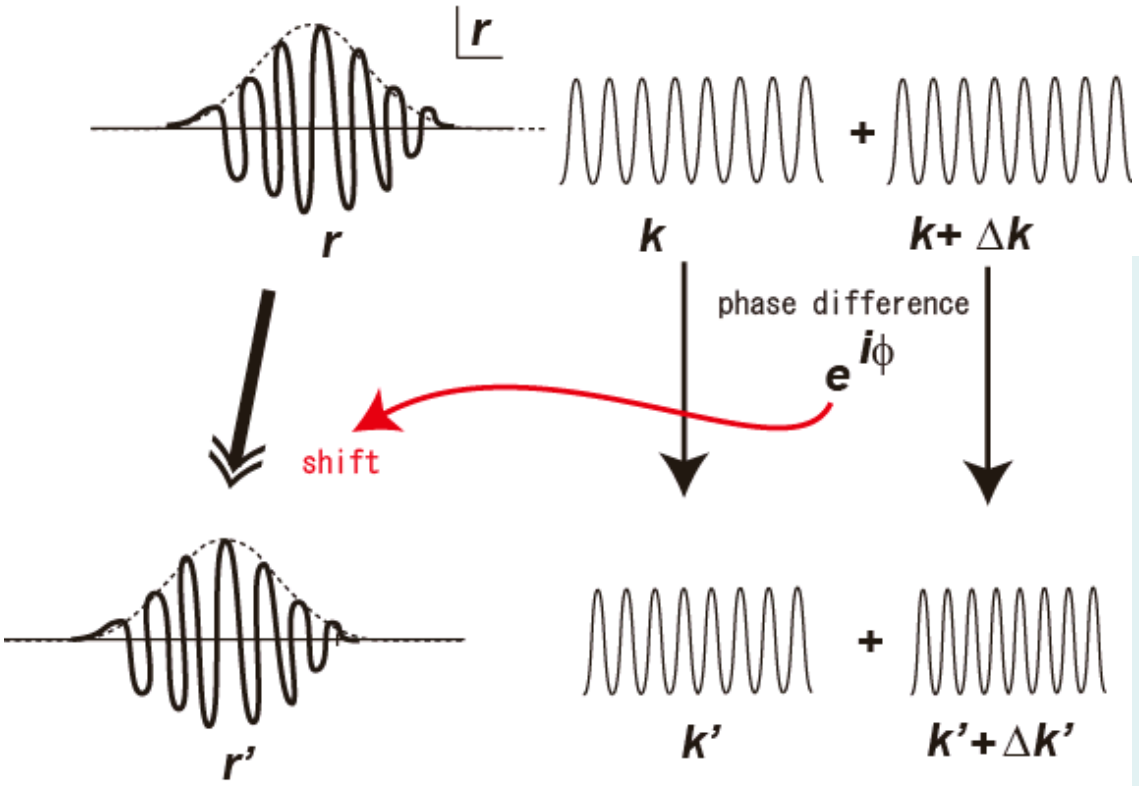
→ Hall effect

$$\vec{\Omega}_n(\vec{k}) = i\left\langle \frac{\partial u_n}{\partial \vec{k}} \right| \times \left| \frac{\partial u_n}{\partial \vec{k}} \right\rangle$$ **: Berry curvature**

$u_{n\vec{k}}$: periodic part of the Bloch wf. $\psi_{n\vec{k}}(\vec{x}) = u_{n\vec{k}}(\vec{x})e^{i\vec{k}\cdot\vec{x}}$

(n : band index)

Wavepacket motion and Berry phase



(Wavepacket)

$$F(x) = \int dk\, f(k) e^{ikx}$$

$$f(k) \propto e^{-\frac{(k-k_c)^2}{a^2}}$$

Force $\vec{F}$ → $\vec{k}$ changes

$$\hbar \dot{\vec{k}} = \vec{F}$$

$f(k)$ obtains a phase during propagation

$$\tilde{f}(k) \propto \exp(-ik\delta)$$

↓

Wavepacket center shifts by δ

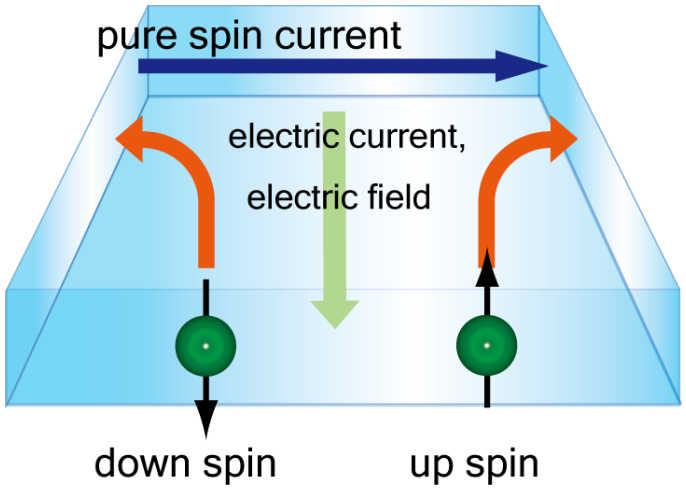
Semiclassical eq. of motion (Adams, Blount; Sundaram,Niu)

$$\dot{\vec{x}} = \frac{1}{\hbar}\frac{\partial E_n(\vec{k})}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Omega}_n(\vec{k})$$

$$\dot{\vec{k}} = -e\vec{E}$$

Example 1:
spin Hall effect of electrons

Intrinsic spin Hall effect in metals& semiconductors



- SM, Nagaosa, Zhang, Science (2003)
- Sinova et al., Phys. Rev. Lett. (2004)

semiclassical eq. of motion for wavepackets

$$\begin{cases} \dot{\vec{x}} = \frac{1}{\hbar}\frac{\partial E_n(\vec{k})}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Omega}_n(\vec{k}) \\ \dot{\vec{k}} = -e\vec{E} \end{cases}$$

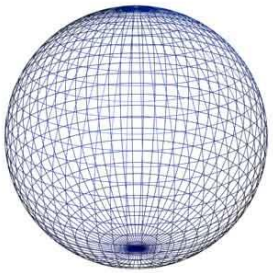
Force // electric field

Adams, Blount; Sundaram,Niu, …

$$\vec{\Omega}_n(\vec{k}) = i\left\langle \frac{\partial u_n}{\partial \vec{k}} \middle| \times \middle| \frac{\partial u_n}{\partial \vec{k}} \right\rangle$$ **: Berry curvature**

$u_{n\vec{k}}$: periodic part of the Bloch wf.

$\psi_{n\vec{k}}(\vec{x}) = u_{n\vec{k}}(\vec{x})e^{i\vec{k}\cdot\vec{x}}$ (n : band index)



Spin-orbit coupling → Berry curvature depends on spin

Example 2:
Hall effect of light

Spin Hall effect of light

Onoda, SM, Nagaosa, Phys. Rev. Lett. (2004)
Onoda, SM, Nagaosa, Phys. Rev. E (2006)

Semiclassical eq. of motion

$$\begin{cases} \dot{\vec{r}} = v(\vec{r})\hat{k} + \dot{\vec{k}} \times \langle z | \vec{\Omega}(\vec{k}) | z \rangle \\ \dot{\vec{k}} = -k\nabla v(\vec{r}) \\ |\dot{z}\rangle = -i\dot{\vec{k}} \cdot \vec{\Lambda}(\vec{k}) | z \rangle \end{cases}$$

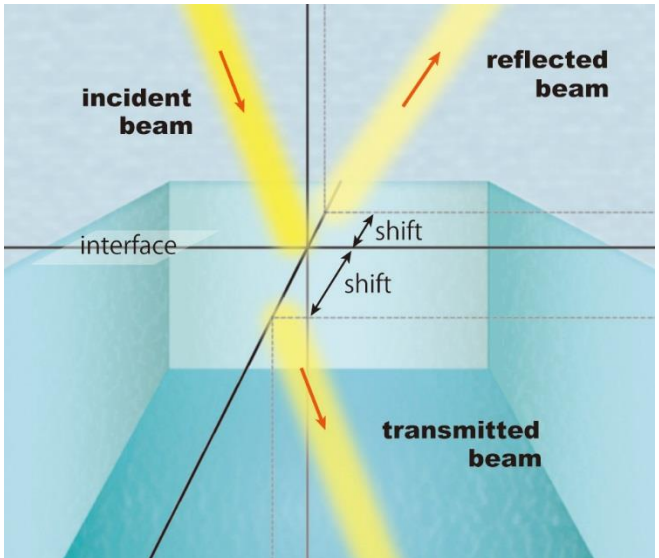
gradient of refractive index
→ spin Hall effect of light"
→ Shift of a trajectory of light beam

Geometrical optics
"Fermat's principle"

$v(\vec{r}) = \dfrac{c}{n(\vec{r})}$: slowly varying
$|z\rangle$: polarization
$\vec{\Lambda}(\vec{k})$: gauge field
$\vec{\Omega}(\vec{k})$: Berry curvature

Imbert shift

Theory: Fedorov (1955)
Experiment: Imbert, PRD (1972)
Hosten, Kwiat, Science (2008)



In the vacuum

$$\vec{\Omega}(\vec{k}) = \frac{\vec{k}}{k^3} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

→ Left circular pol.
→ right

← spin-orbit coupling of light
(transverse only)

Imbert shift of light beam = Berry curvature of photon

Semiclassical eq. of motion

$$\dot{\vec{r}} = v(\vec{r})\frac{\vec{k}}{k} + \dot{\vec{k}} \times (z \,|\, \vec{\Omega}_{\vec{k}} \,|\, z) \qquad \dot{\vec{k}} = -k\nabla v(\vec{r})$$

Berry phase

$$\vec{\Omega}(\vec{k}) = \frac{\vec{k}}{k^3}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

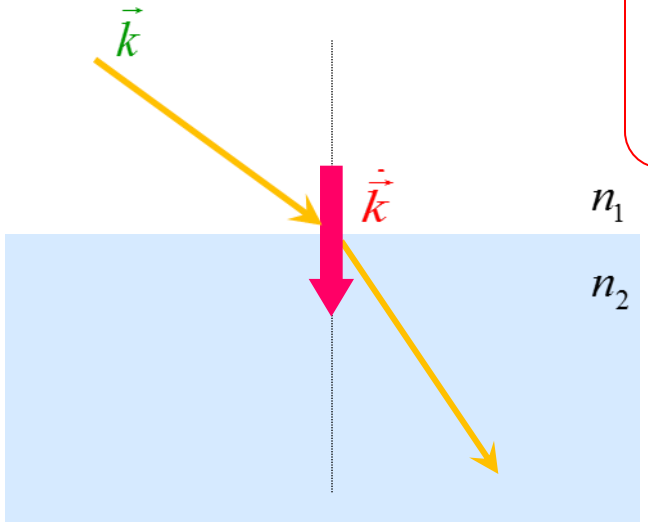
→ Left circular pol. → Anomalous velocity = ⊙

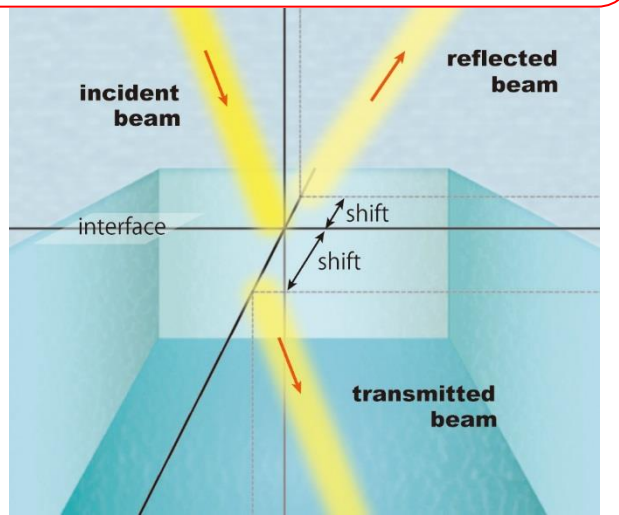
→ Right circular pol. → Anomalous velocity = ⊗

For refracted light, ⊙ ⊗ = transverse shift

Imbert shift

Theory: Fedorov (1955)
Experiment: Imbert, PRD (1972)
Hosten, Kwiat, Science (2008)





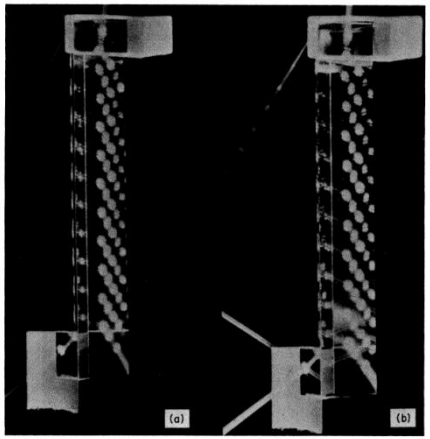
Onoda, SM, Nagaosa, Phys. Rev. Lett. (2004)
Onoda, SM, Nagaosa, Phys. Rev. E (2006)

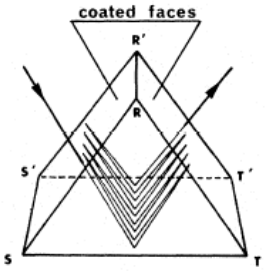
Experiments on imbert shift : Shift of light beam in reflection/refraction

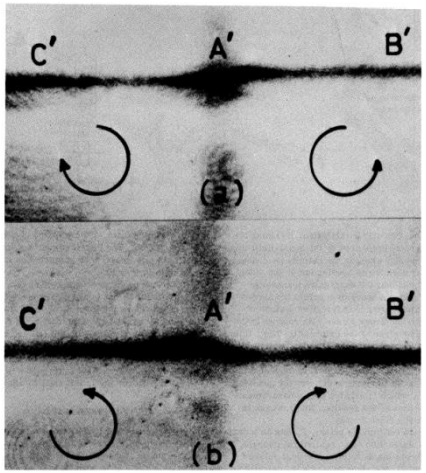
Magnitude of the shift $\approx \lambda$
Width of the beam is much larger → not easy to observe.

total reflection

Imbert, Phys. Rev. D5, 787 (1972)



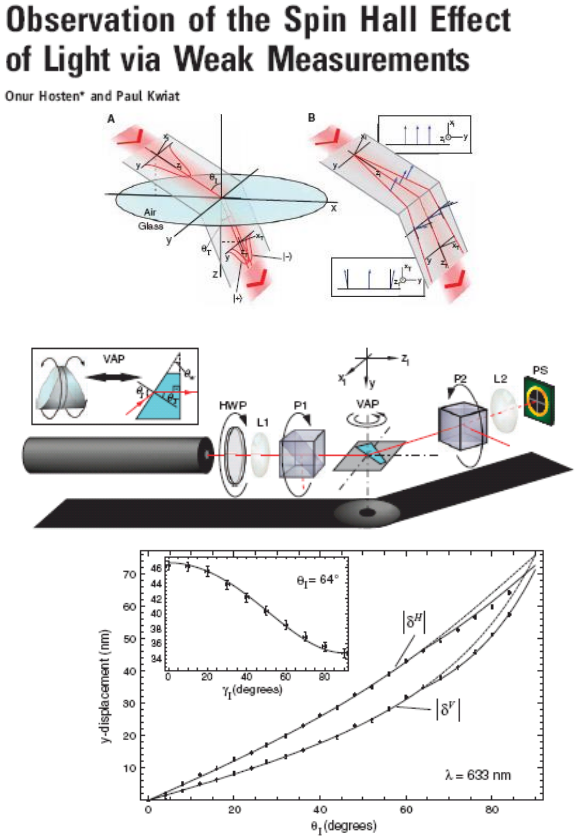




28 total reflections → shift is enhanced

refraction

Hosten, Kwiat, Science 319,787 (2008)



Good agreement with theory

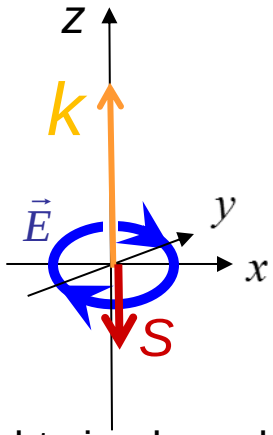
Spin Hall effect of light

strong spin orbit coupling in photon →→→ nonzero Berry curvature

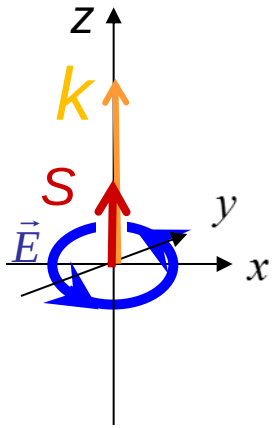
k and S are either
parallel or antiparallel.

$$\vec{\Omega}(\vec{k}) = \frac{\vec{k}}{k^3}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

→ Left circular
→ Right circular



right circular pol.
k and S antiparallel



left circular pol.
k and S parallel

Example 3:
Magnon thermal Hall effect

- Matsumoto, Murakami, Phys. Rev. Lett. 106, 197202 (2011).
- Matsumoto, Murakami, Phys. Rev. B 84, 184406 (2011)

From electrons to magnons (spin waves)

- electron: : charge=-e, fermion
- Spin wave (magnon): charge=0, boson



wave = represented by Bloch wavefunction

electrons

$$\begin{cases} \dot{\vec{x}} = \frac{1}{\hbar}\frac{\partial E_n(\vec{k})}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Omega}_n(\vec{k}) \\ \dot{\vec{k}} = -e\left(\vec{E} + \dot{\vec{x}} \times \vec{B}\right) \end{cases}$$

$$\vec{\Omega}_n(\vec{k}) = i\left\langle \frac{\partial u_n}{\partial \vec{k}} \middle| \times \middle| \frac{\partial u_n}{\partial \vec{k}} \right\rangle$$

: Berry curvature

magnons

$$\begin{cases} \dot{\vec{x}} = \frac{1}{\hbar}\frac{\partial E_n(\vec{k})}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Omega}_n(\vec{k}) \\ \dot{\vec{k}} = -\nabla V \end{cases}$$

V(r): confinement potential

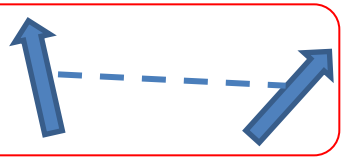
- Exchange magnons (quantum-mechanical)
  e.g. $Lu_2V_2O_7$
  (Katsura et al. (2011), Onose et al.(2011))
- Magnetostatic spin waves (classical)
  e.g. YIG (yttrium iron garnet)
  R. Matsumoto, S. Murakami,
  Phys. Rev. Lett. 106, 197202 (2011).

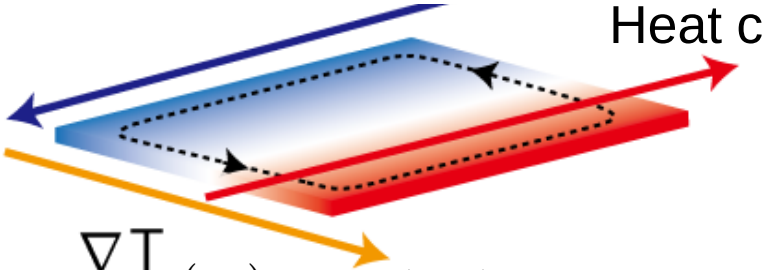
Magnon Thermal Hall effect in ferromagnet

Magnon (spin wave) = low-energy excitations in magnetts



Dipolar interaction → spin-orbit coupling



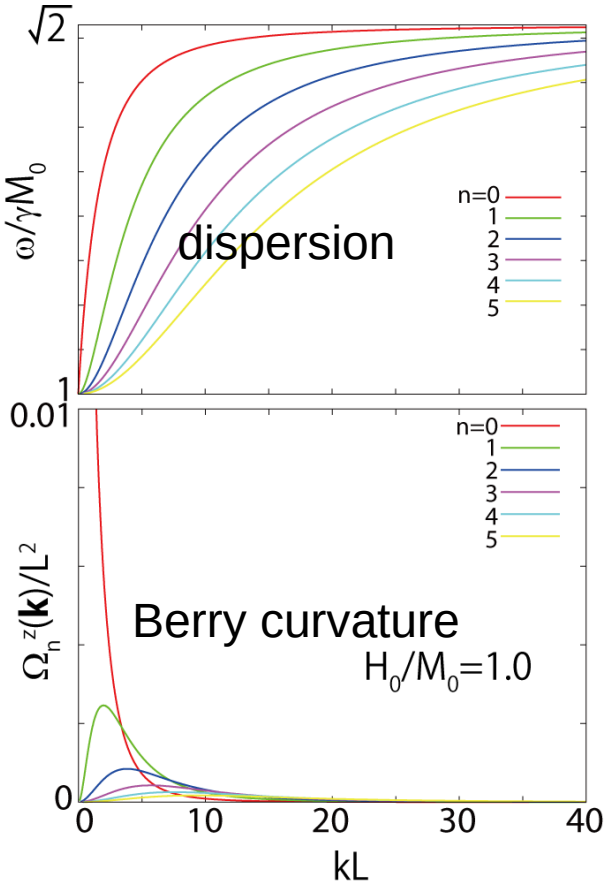


$$\left(j_Q\right)_x = \kappa_{xy}\left(\nabla T\right)_y$$

Thermal Hall conductivity

$$\kappa_{xy} = -\frac{2k_B^2 T}{\hbar V}\sum_{n,k} c_2\left(\rho(\varepsilon_{nk})\right)\Omega_n^z\left(\vec{k}\right)$$

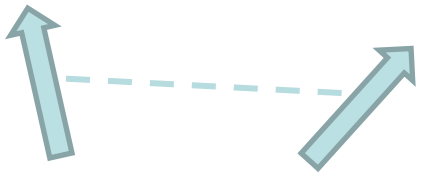
$$c_2(\rho) = \int_0^\rho \left[\log\left(\frac{1+t}{t}\right)\right]^2 dt = (1+\rho)\left[\log\left(\frac{1+\rho}{\rho}\right)\right]^2 - (\log\rho)^2 - 2\mathrm{Li}_2(-\rho)$$



Matsumoto, Murakami, Phys. Rev. Lett. 106, 197202 (2011).

Matsumoto, Murakami, Phys. Rev. B 84, 184406 (2011)

Magnetic dipole interaction

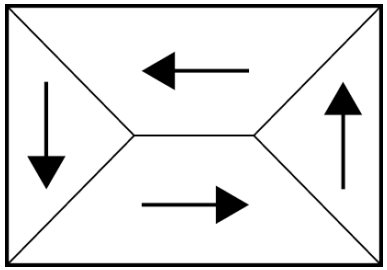


$$H_{\text{dipole}} = \frac{\mu_0}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|^3}\left\{3\,\frac{\boldsymbol{S}_{\boldsymbol{r}}\cdot(\boldsymbol{r}-\boldsymbol{r}')\;\boldsymbol{S}_{\boldsymbol{r}'}\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2}-\boldsymbol{S}_{\boldsymbol{r}}\cdot\boldsymbol{S}_{\boldsymbol{r}'}\right\}.$$

- Dominant in long length scale (microns)
- Similar to spin-orbit int.

  → Berry curvature

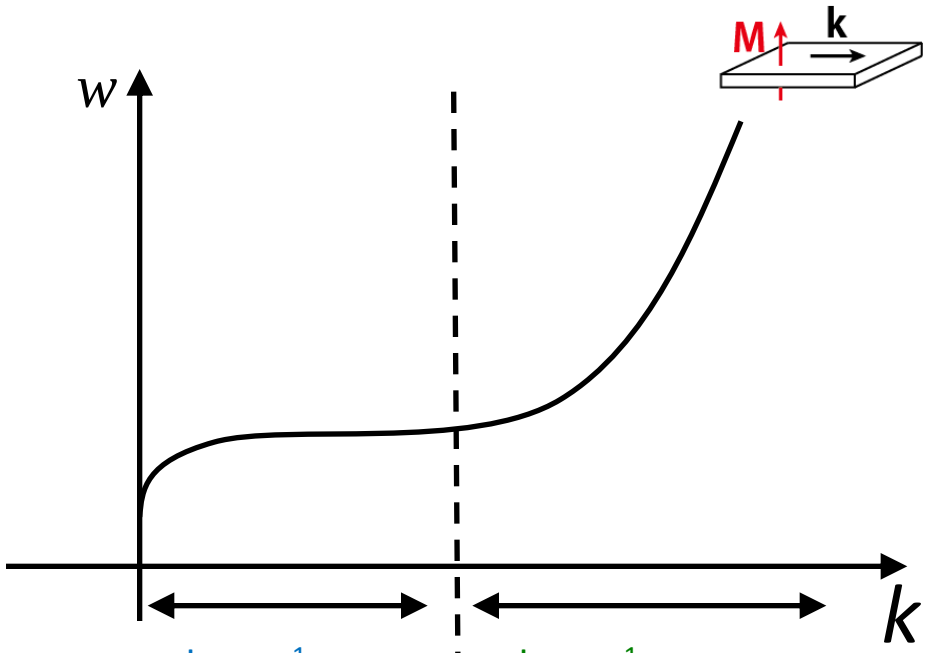
- Long-ranged → nontrivial, controlled by shape



Magnetic domains

Spin wave dispersion for thin-film ferromagnet

MagnetoStatic Forward Volume Wave



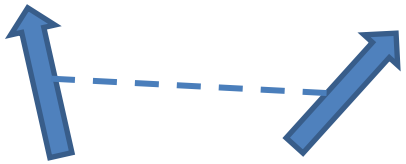
k~μm^{-1}

~GHz

Magnetostatic mode

← dipolar interaction

Classical, long-ranged



→ "spin-orbit coupling"

→ nonzero Berry curvature

k~nm^{-1}

GHz~THz

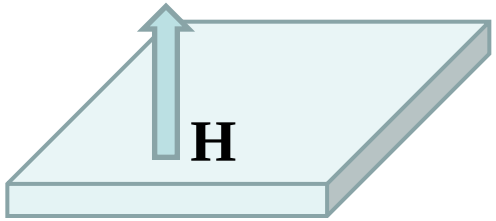
Magnon

← Exchange $E \propto k^2$

quantum mechnical

short ranged

Magnetostatic modes in ferromagnetic films (YIG)



- Landau-Lifshitz (LL) equation $\frac{d\mathbf{M}}{dt} = -\gamma(\mathbf{M}\times\mathbf{H})$
- Maxwell equation $\nabla\cdot\mathbf{B} = 0 \quad , \quad \nabla\times\mathbf{H} = 0$
- Boundary conditions $\mathbf{B}_{1\perp} = \mathbf{B}_{2\perp} \quad , \quad \mathbf{H}_{1//} = \mathbf{H}_{2//}$

(Only the dipolar coupling is considered.)

Generalized eigenvalue eq.

B. A. Kalinikos and A. N. Slavin, *J. Phys. C* **19**, 7013 (1986)

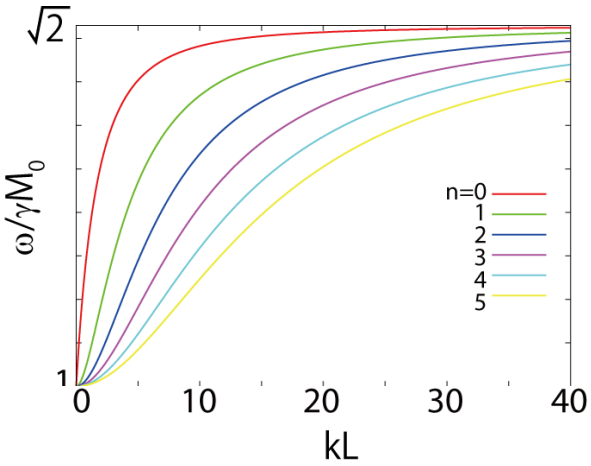
$$\hat{H}\mathbf{m}(z) = \omega\sigma_z\mathbf{m}(z) \qquad \left(\hat{H}\mathbf{m}(z) \equiv \omega_H\mathbf{m}(z) - \omega_M\int_{-L/2}^{L/2} dz'\hat{G}(z,z')\mathbf{m}(z')\right)$$

$w_H = gH_0, \quad w_M = gM_0, \quad L$: thickness of the film, $s_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{m}(z) = \begin{pmatrix} m_x + im_y \\ m_x - im_y \end{pmatrix}$

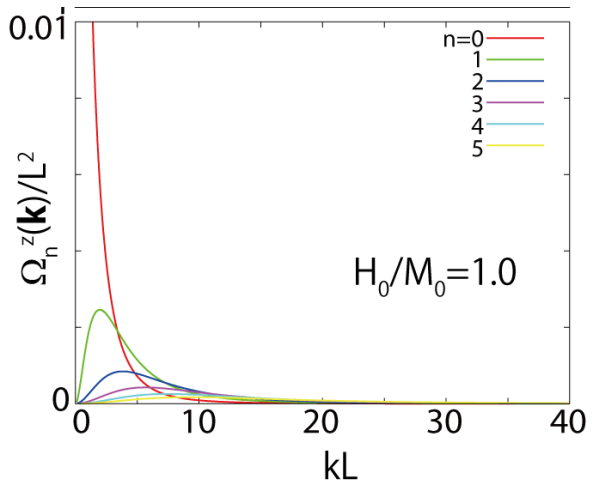
Berry curvature

$$\Omega_n^{\gamma}(\mathbf{k}) = -\varepsilon_{\alpha\beta\gamma}\,\mathrm{Im}\left\langle \frac{\partial\mathbf{m}_{n,\mathbf{k}}}{\partial k_\alpha}\middle|\sigma_z\middle|\frac{\partial\mathbf{m}_{n,\mathbf{k}}}{\partial k_\beta}\right\rangle$$

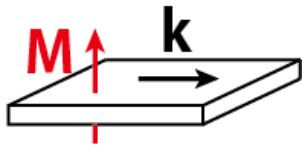
Berry curvature
for magnetostatic forward volume wave mode



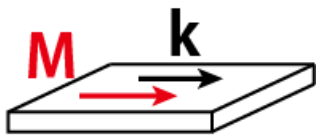
: Band structure for magnetostatic forward volume-wave mode

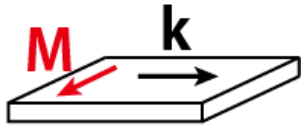


: Berry curvature



Berry curvature is zero for backward volume wave and surface wave





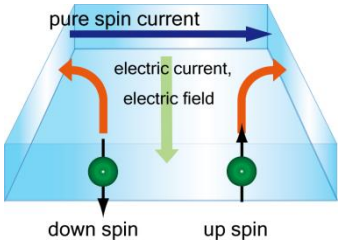
- R. Matsumoto, S. Murakami, PRL 106,197202 (2011), PRB84, 184406 (2011)

Phenomena due to Berry curvature of band structure

Gapless

Exert a force → transverse motion

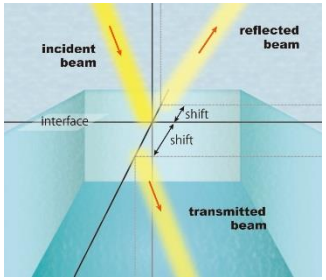
- Hall effect — Electric field
- Spin Hall effect (of electrons) — Electric field



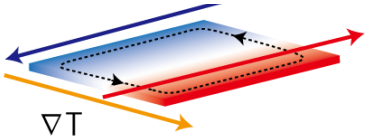
Fermions

Bosons

- Spin Hall effect of light — Spatial gradient of refractive index



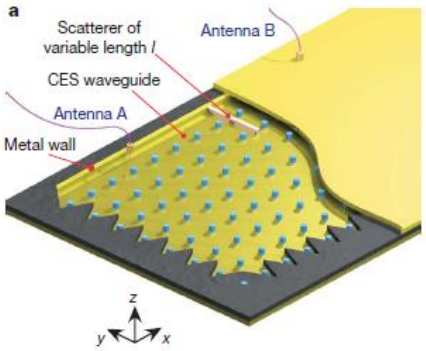
- Magnon thermal Hall effect — Temperature gradient

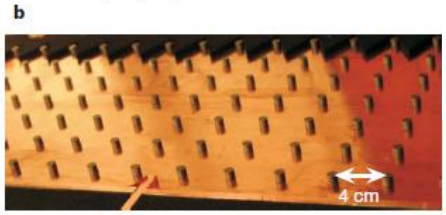


§ 4 Topological plasmonic crystals

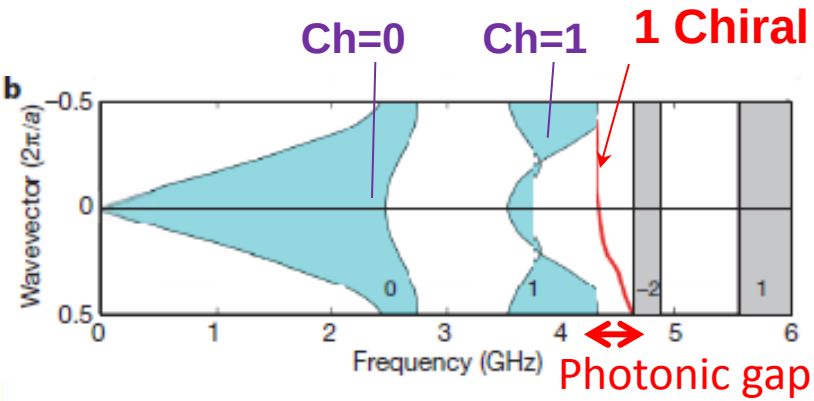
Optical one-way waveguide in photonic crystal: -- Topological photonic crystal --

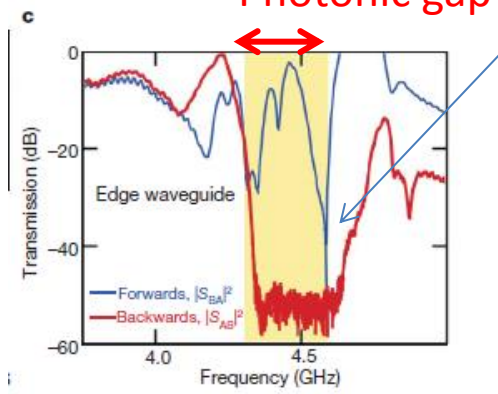
Theory: Haldane, Raghu, PRL100, 013904 (2008)
Experiment: Wang, Chong, Joannopoulos, Soljačić, Nature 461, 772 (2009)

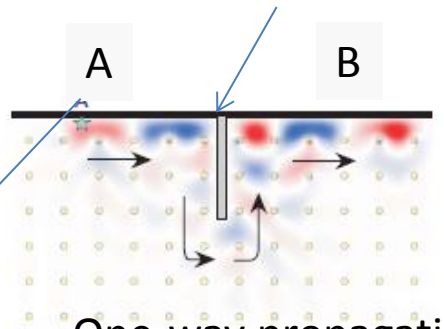




Periodic array of ferrite rods in air
(Periodicity =4cm)







One-way propagation
(10^5 times difference)

(A→ B)/(B→A) =10^5
Chiral mode !

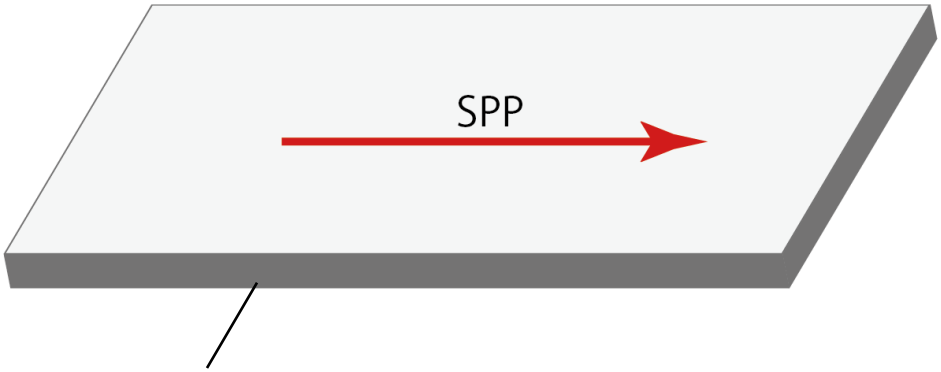
Surface plasmon polariton(SPP)

Electromagnetic modes propagating
on a metal surface with negative dielectric constant

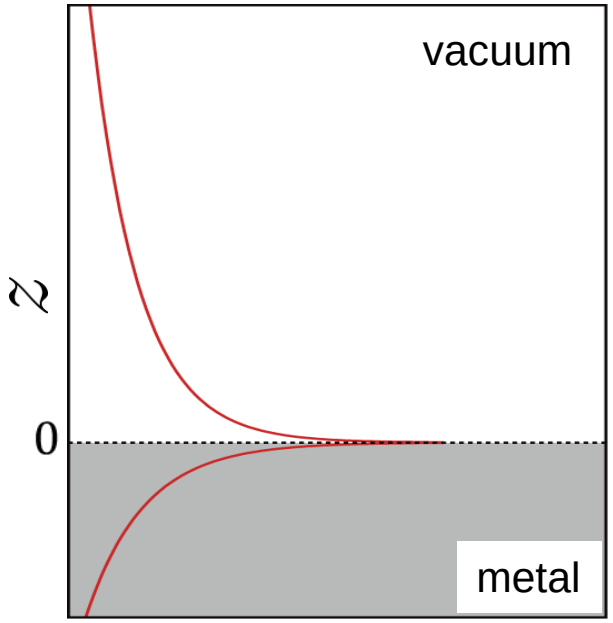
Dielectric constant $\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$

ω_p : plasma frequency

→ It becomes negative for $\omega < \omega_p$



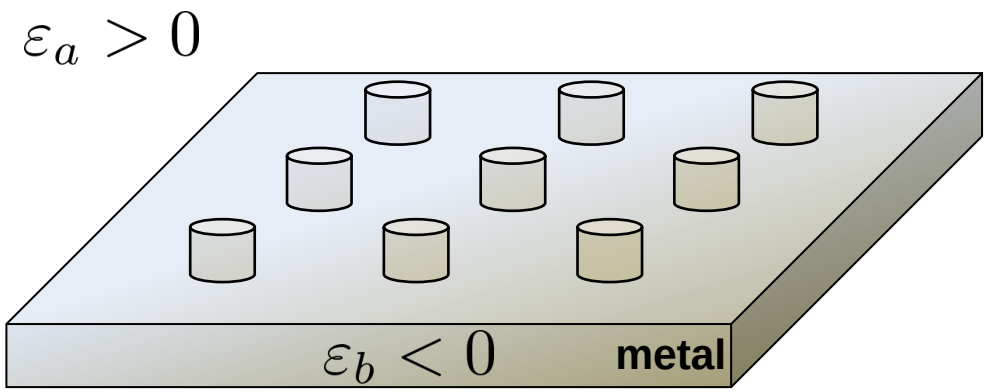
Metal with negative dielectric constant

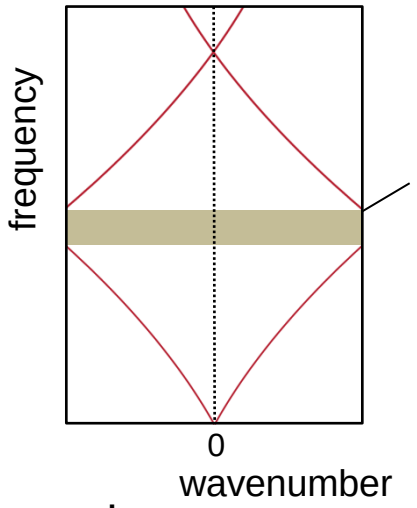


Intensity

Topological plasmonic crystals??

" quantum Hall effect for plasmons by introducing spatial periodicity





- We need to formulate Berry curvature and Chern number

- Hermitian formulation is required

Previous work

$$-i\frac{d\zeta(u)}{du} = \boldsymbol{T}\zeta(u) \qquad \boldsymbol{T} = \begin{pmatrix} KD_0 & -KD_2 & C_0 & C_2 \\ KD_{-2} & -KD_0 & C_{-2} & C_0 \\ -K^2C_0 + \tilde{\varepsilon}\frac{\omega^2}{c_0^2} & KC_2 & KD_0 & KC_2 \\ KC_{-2} & -K^2C_0 + \tilde{\varepsilon}\frac{\omega^2}{c_0^2} & -KD_{-2} & -KD_0 \end{pmatrix}$$

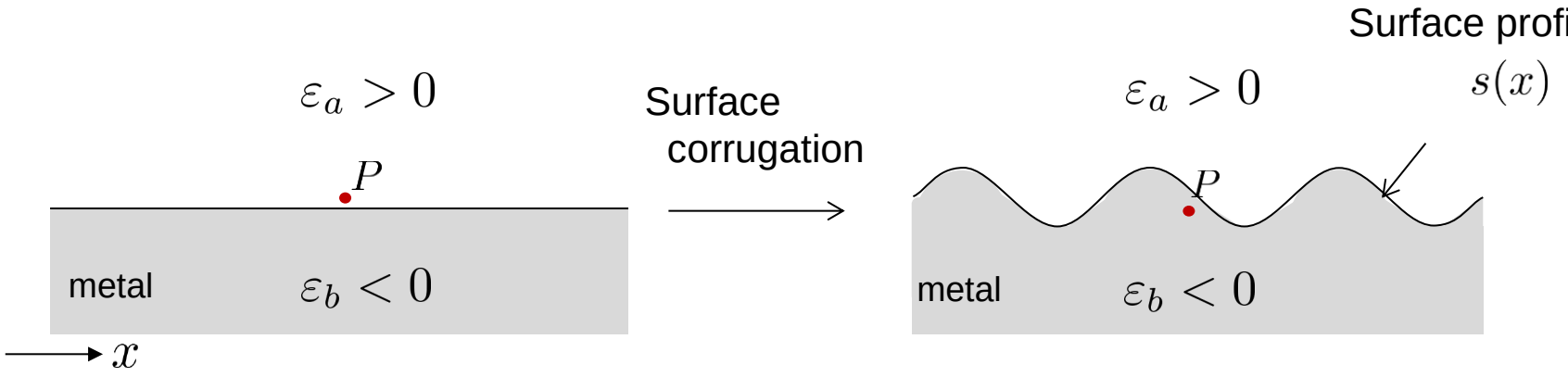
Non-hermitian $\longrightarrow$

- Formulation of perturbation theory is not easy.
- Eigenfrequency may become complex

Towards reformulation of SPP equations…:

For calculation of Berry curvature:
we need to calculate wavefunctions analytically

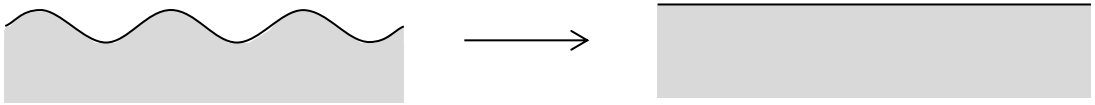
→ we focus on small surface corrugation & apply perturbation theory



Even for small s(x), the change of the dielectric constant at the point P is large

→ perturbation theory cannot be applied in the usual way.

→ We introduce a coordinate transformation

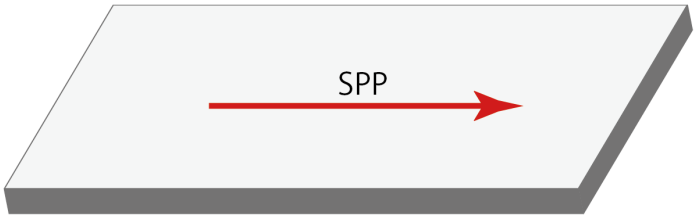


Hermitian eigenvalue equation for 1D plasmonic crystals

Maxwell eq. for flat surface

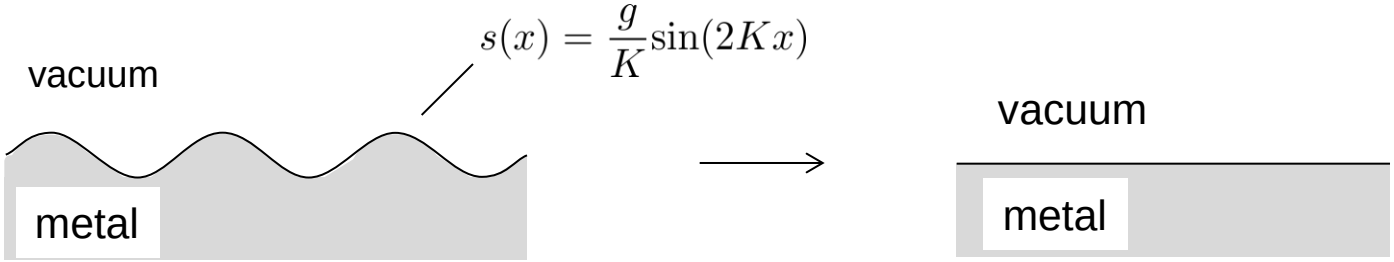
Kitamura, Murakami, PRB 88, 045406 (2013)

$$\frac{1}{\sqrt{\varepsilon\mu}}\begin{pmatrix} 0 & 0 & -i\partial_z \\ 0 & 0 & i\partial_x \\ -i\partial_z & i\partial_x & 0 \end{pmatrix}\psi = \omega\psi$$



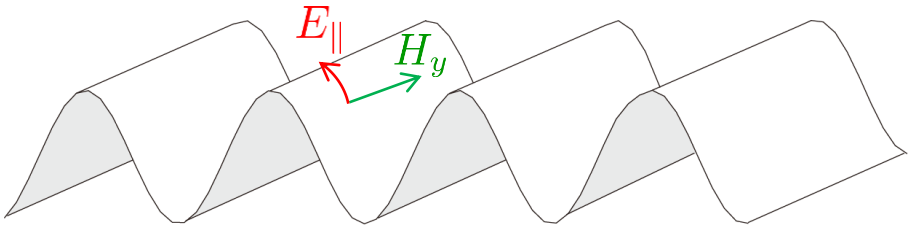
Eigenvectors for unperturbed problem

Include surface corrugation as a perturbation: coordinate transformation



（For simplicity the corrugation is set to be sinusoidal）

Boundary condition for electromagnetic wave



- Boundary condition for $E_{//}$
- Boundary condition for $H_{//}$

Hermitian eigenvalue problem for SPP

$$\begin{pmatrix} i\left(\frac{\lambda_0^a}{\varepsilon_a} - \frac{\lambda_0^b}{\varepsilon_b}\right) & -iu(k)g \\ iu(k)g & i\left(\frac{\lambda_{-1}^a}{\varepsilon_a} - \frac{\lambda_{-1}^b}{\varepsilon_b}\right) \end{pmatrix}\varphi = 0$$

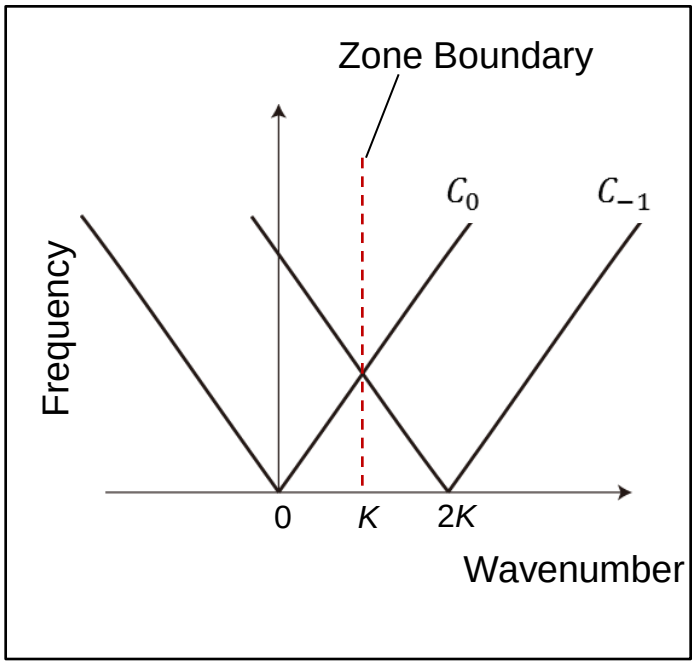
λ^a: Wavenumber in vacuum
λ^b: Wavenumber in metal
ε^a: Dielectric const. in vacuum
ε^b: Dielectric const. in metal

Eigenvalue eq. for SPP

Kitamura, Murakami, PRB 88, 045406 (2013)

$$\begin{pmatrix} i\left(\frac{\lambda_0^a}{\varepsilon_a}-\frac{\lambda_0^b}{\varepsilon_b}\right) & -iu(k)g \\ iu(k)g & i\left(\frac{\lambda_{-1}^a}{\varepsilon_a}-\frac{\lambda_{-1}^b}{\varepsilon_b}\right) \end{pmatrix} \varphi = 0$$

$$\varphi = \begin{pmatrix} C_0 \\ C_{-1} \end{pmatrix}$$ (Amplitudes for two waves)



Hermiticity

Hermitian when λ_a, λ_b are pure imaginary

- In the metal

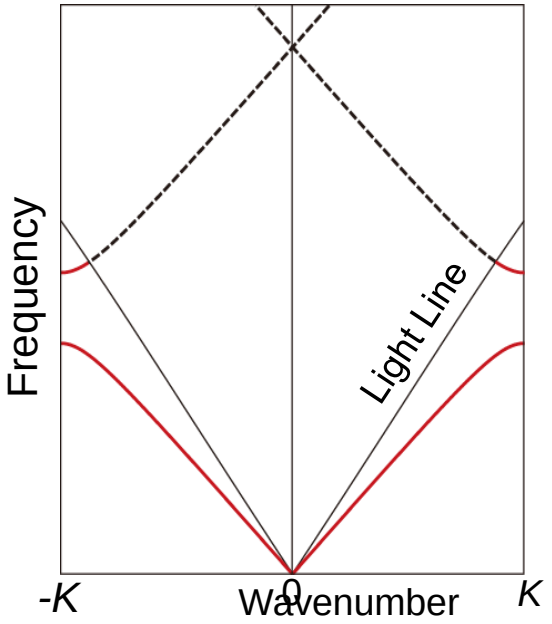
$$(\lambda_m^b)^2 = \varepsilon_b \mu_0 \omega_{\mathrm{SPP}}^2 - (k-2mK)^2$$

$\longrightarrow$ Automatically satisfied by $\varepsilon_b < 0$

- In vacuum

$$(\lambda_m^a)^2 = \varepsilon_a \mu_0 \omega_{\mathrm{SPP}}^2 - (k-2mK)^2$$

$\longrightarrow$ Satisfied when $\omega_{\mathrm{SPP}} < \omega_{\mathrm{Light}}$



Outside of the light cone: Hermitian
Inside of the light cone: non-Hermitian

(radiation of the SPP into vacuum)

At Brillouin zone boundary

- eigenfrequency

$$\omega_{\pm}^2 = \frac{K^2}{\mu_0}\left(\frac{1}{\varepsilon_a}+\frac{1}{\varepsilon_b} \pm 2g\sqrt{\frac{1}{|\varepsilon_a\varepsilon_b|}}\right)$$

K : wavenumber at zone boundary

g : size of corrugation

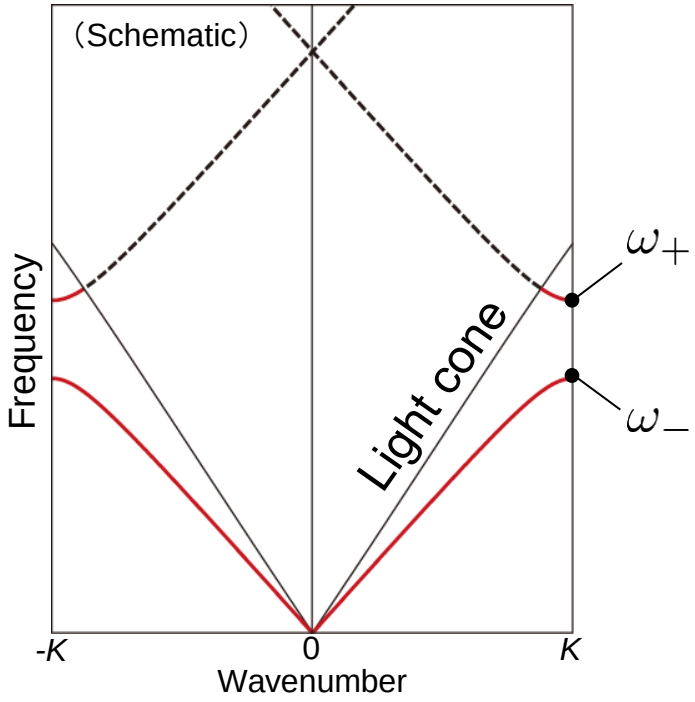
- eigenfunction

$$\varphi = \begin{pmatrix} 1 \\ \pm i e^{i\phi} \end{pmatrix} \approx \begin{pmatrix} C_0^{\alpha} \\ C_{-1}^{\alpha} \end{pmatrix} \longrightarrow$$

$$E_x \propto \sqrt{\frac{1 \mp \sin(2Kx+\phi)}{2}}\cos(\omega t - \phi/2)e^{-i\lambda_\alpha z}$$

$$E_z \propto \sqrt{\frac{1 \pm \sin(2Kx+\phi)}{2}}\cos(\omega t - \phi/2)e^{-i\lambda_\alpha z}$$

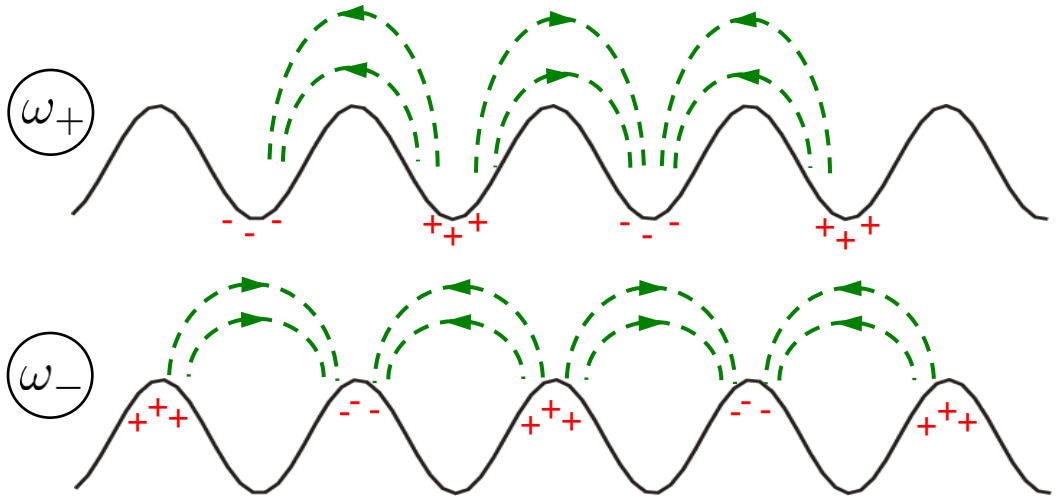
$$H_y \propto \sqrt{\frac{1 \mp \sin(2Kx+\phi)}{2}}\cos(\omega t - \phi/2)e^{-i\lambda_\alpha z}$$



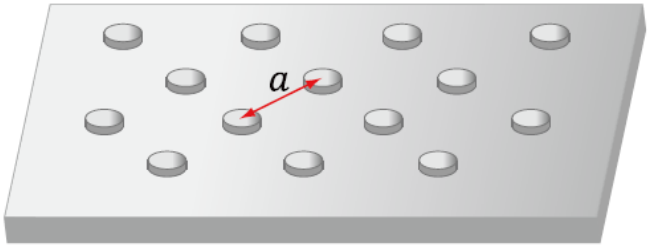
Surface corrugation

$$s(x) = -\frac{g}{2iK}(e^{-2iKx-i\phi} - e^{2iKx+i\phi})$$

Band gap at the zone boundary ***E***



Two-dimensional plasmonic crystal

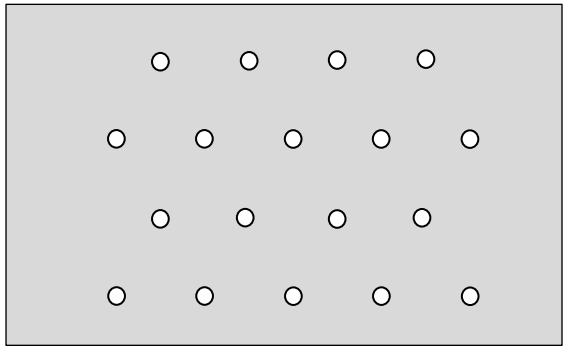


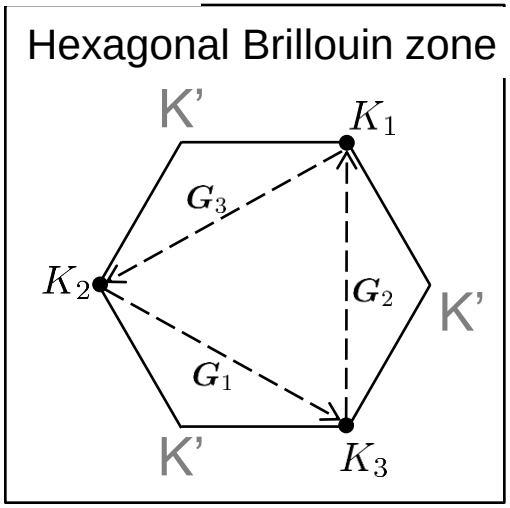
Solution for flat surface

$$\psi(\boldsymbol{k}) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \frac{c}{\sqrt{2}k\omega} \begin{pmatrix} -\lambda k_x \\ -\lambda k_y \\ k^2 \\ -\frac{\omega}{c} k_y \\ \frac{\omega}{c} k_x \\ 0 \end{pmatrix} \mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{r}+\lambda z)}.$$

Corrugation forming triangular lattice → Dirac cones at the Brillouin zone corners.

Surface corrugation $s(\boldsymbol{r}) = d\sum_{i=1}^{3} \cos(\boldsymbol{G}_i \cdot \boldsymbol{r})$ G_i: Reciprocal lattice vector





Boundary condition at K point

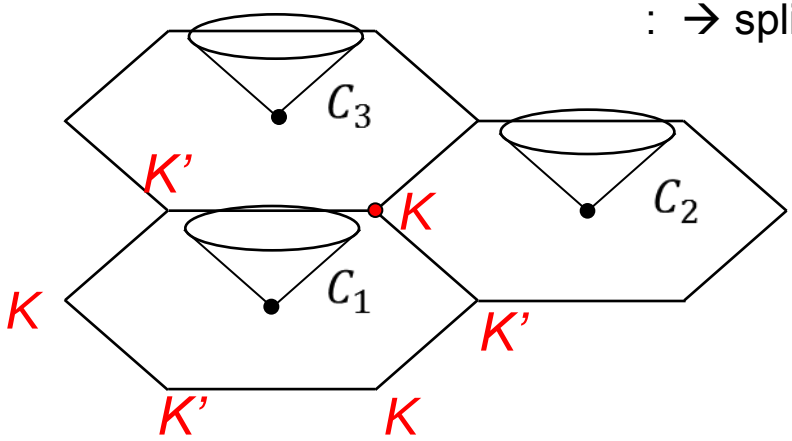
$$\begin{pmatrix} -K_{1x}\left(\frac{\lambda_a}{\varepsilon_a} - \frac{\lambda_b}{\varepsilon_b}\right) & -i\frac{K^2 d}{2} K_{1x}\left(\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_b}\right) & -i\frac{K^2 d}{2} K_{1x}\left(\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_b}\right) \\ -i\frac{K^2 d}{2} K_{2x}\left(\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_b}\right) & -K_{2x}\left(\frac{\lambda_a}{\varepsilon_a} - \frac{\lambda_b}{\varepsilon_b}\right) & -i\frac{K^2 d}{2} K_{2x}\left(\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_b}\right) \\ -i\frac{K^2 d}{2} K_{3x}\left(\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_b}\right) & -i\frac{K^2 d}{2} K_{3x}\left(\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon_b}\right) & -K_{3x}\left(\frac{\lambda_a}{\varepsilon_a} - \frac{\lambda_b}{\varepsilon_b}\right) \end{pmatrix} \Psi = 0$$

Dirac cones at K and K' → gap opening by time-reversal symm. breaking

Solutions at the zone corners (K,K points)

Solutions at flat surface → hybridization due to surface corrugation:

: → splits into doublet & singlet at K & K'



$$\Psi^{\alpha} = \sum_{i=1}^{3} C_i^{\alpha} \psi^{\alpha}(\boldsymbol{K}_i)$$

Singlet

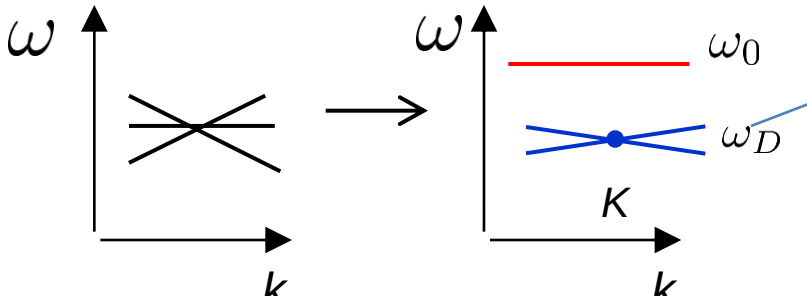
$$\Psi_1 = \psi(\boldsymbol{K}_1) + \psi(\boldsymbol{K}_2) + \psi(\boldsymbol{K}_3) \longrightarrow \omega_0^2 = \frac{K^2}{\mu_0}\left(\frac{1}{\varepsilon_a} + \frac{1}{\varepsilon_b}\right) + \frac{2K^3 d}{\mu_0}\sqrt{\frac{1}{|\varepsilon_a \varepsilon_b|}}$$

Doublet

$$\Psi_2 = \psi(\boldsymbol{K}_1) + \eta\psi(\boldsymbol{K}_2) + \eta^*\psi(\boldsymbol{K}_3)$$

$$\Psi_3 = \psi(\boldsymbol{K}_1) + \eta^*\psi(\boldsymbol{K}_2) + \eta\psi(\boldsymbol{K}_3)$$

$$\longrightarrow \omega_D^2 = \frac{K^2}{\mu_0}\left(\frac{1}{\varepsilon_a} + \frac{1}{\varepsilon_b}\right) - \frac{K^3 d}{\mu_0}\sqrt{\frac{1}{|\varepsilon_a \varepsilon_b|}}$$

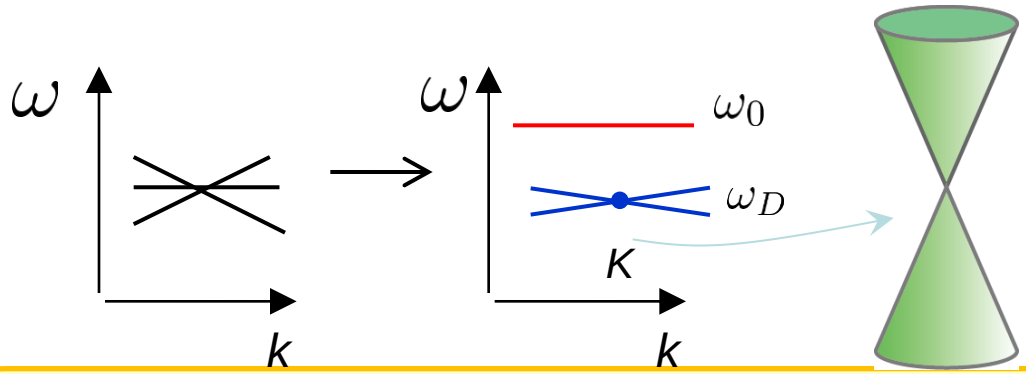


Dirac cone

Group velocity » $\frac{v}{2}$

(v : SPP velocity for flat surface

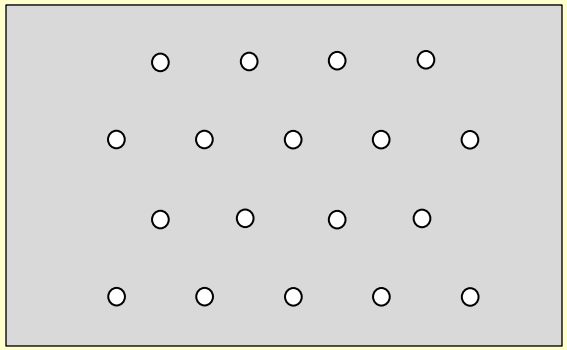
Dirac cones at K and K' → gap opening by time-reversal symm. breaking

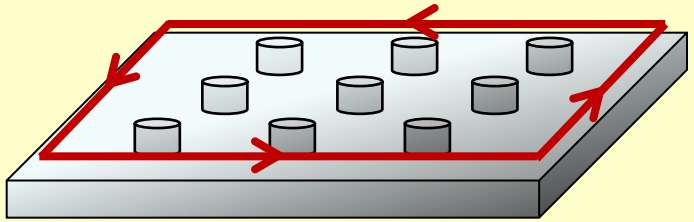


Group velocity » $\frac{v}{2}$

(v: velocity of SPP at flat surface)

- Dirac cone at K and K'
  → because of zero gap, time-reversal symmetry breaking will open a gap
  → topological plasmonic crystal
  lowest band has a nonzero topological number (Chern number)=+1 or -1



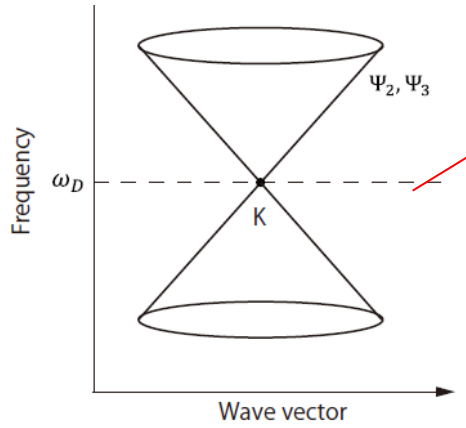


Breaking time-reversal symmetry
→ gap at zone boundaries(K,K')

Magnetic field → off-diagonal components for dielectric tensor

$$\tilde{\varepsilon}_b = \begin{pmatrix} \varepsilon_b & i\frac{\omega_p^2}{\omega^2}\frac{\omega_c\omega}{\omega^2-\omega_c^2} & 0 \\ -i\frac{\omega_p^2}{\omega^2}\frac{\omega_c\omega}{\omega^2-\omega_c^2} & \varepsilon_b & 0 \\ 0 & 0 & \varepsilon_b \end{pmatrix}$$

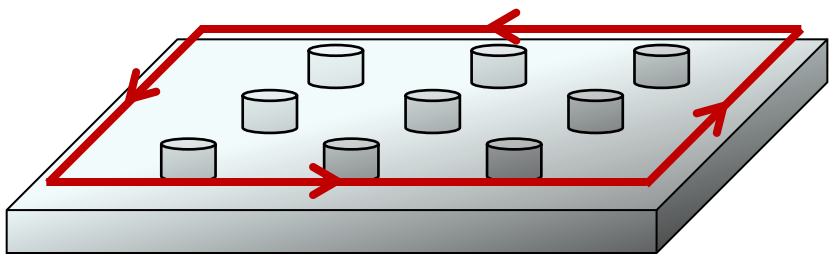
Gap opens at the Dirac cones in K, K'



$$\delta\omega_m = \pm\frac{3\varepsilon_m}{8(\varepsilon_a - \varepsilon_b)}\omega_D.$$

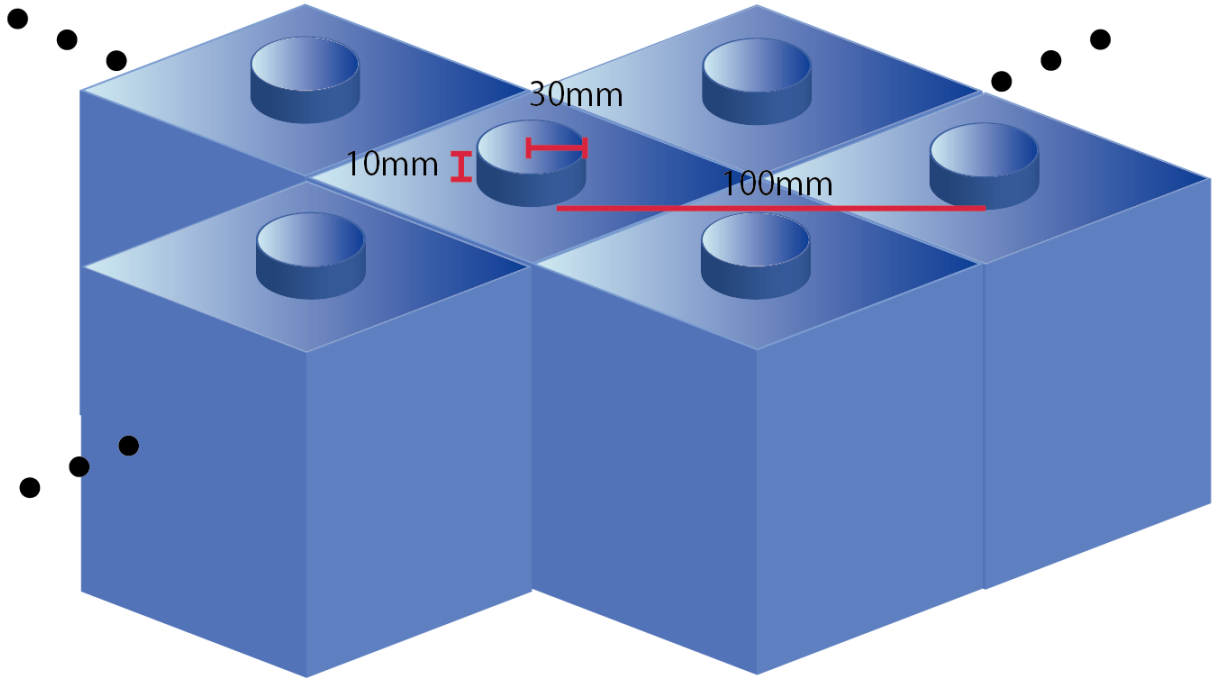
Topological plasmonic crystals

The l owest band has a nontrivial topological number (Chern number =+1)

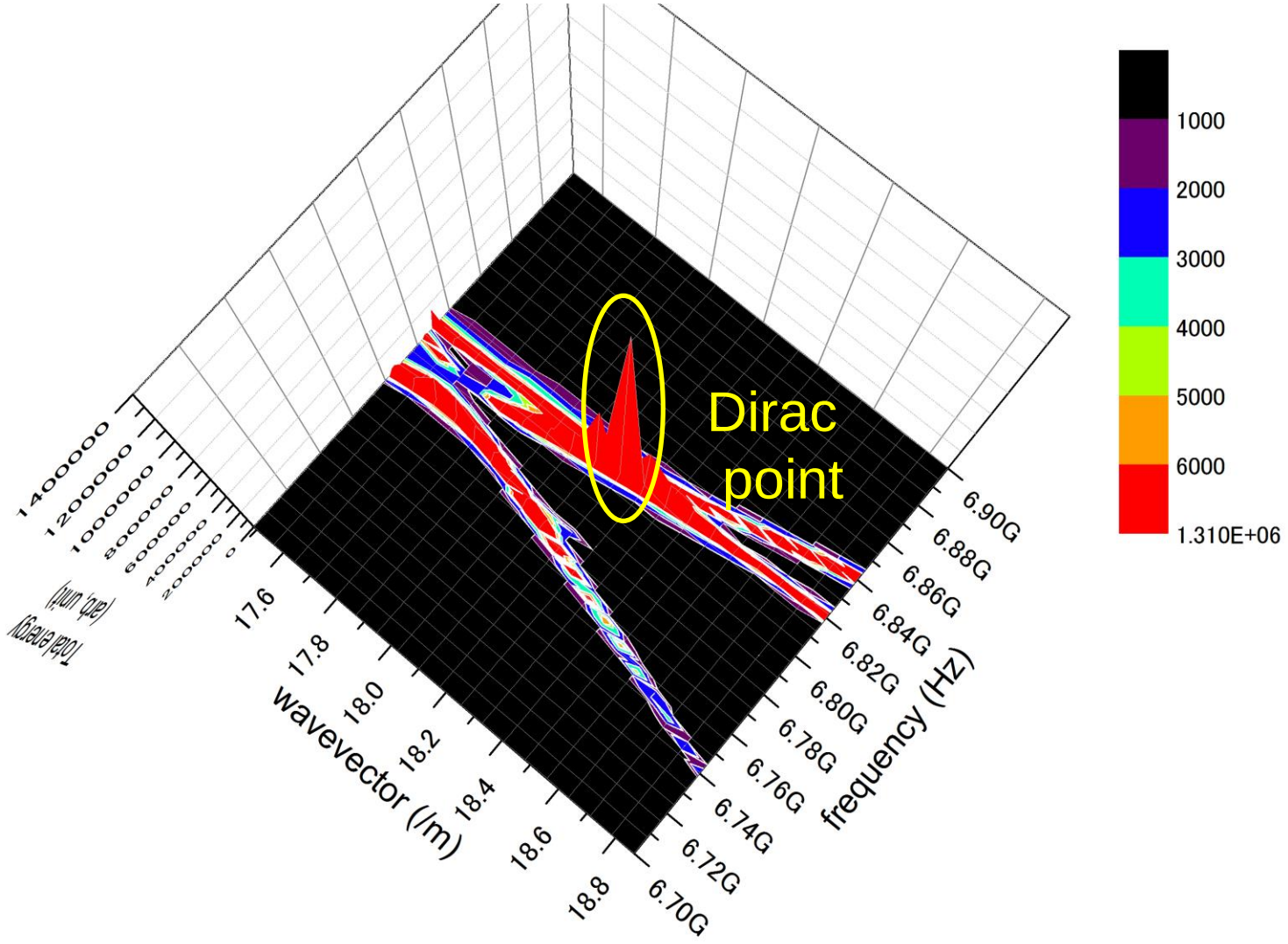


Simulation （multiphysics, COMSOL)

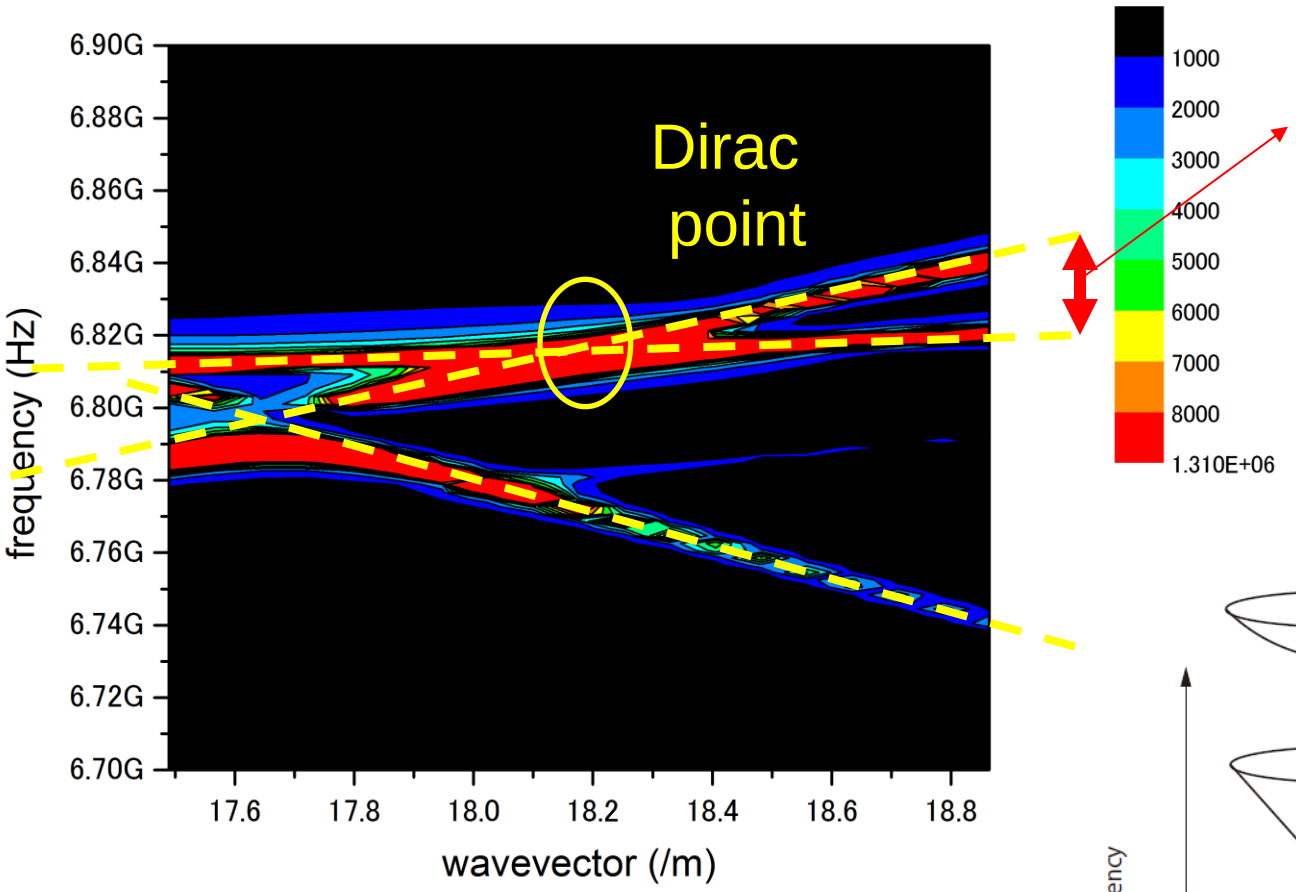
Surface corrugation forming a triangular lattice



Result of the simulation



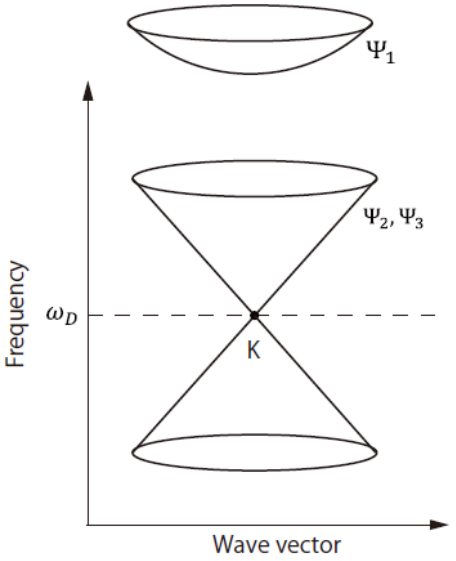
Result of the simulation



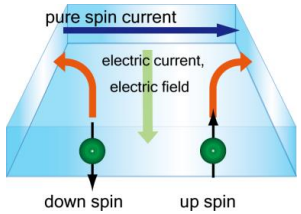
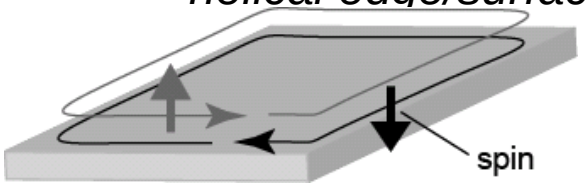
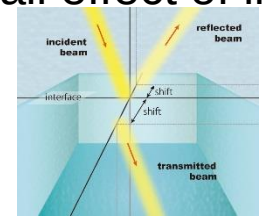
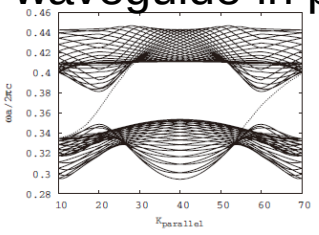
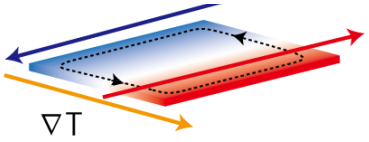
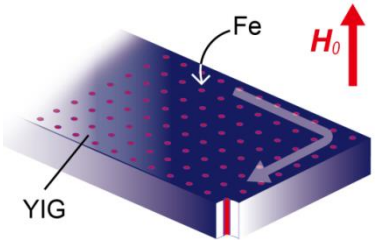
Linear splitting

velocity～$2*10^8$ m/s

～plasmon velocity for flat surface



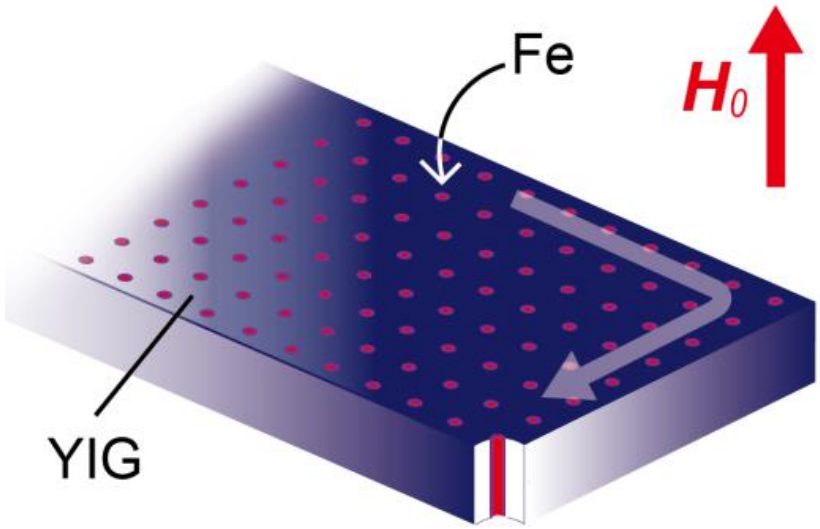
Phenomena due to Berry curvature of band structure

	Gapless — Hall effect	Gapped — topological phase edge/surface modes
Electrons	• Hall effect	→ Quantum Hall effect *chiral edge modes*
Electrons	• Spin Hall effect (of electrons) 	→ Topological insulators *helical edge/surface modes* 
Photons	•Spin Hall effect of light 	→ one-way waveguide in photonic crystal 
Magnons	• Magnon thermal Hall effect 	→ topological magnonic crystal 
Plasmons	• plasmon Hall effect ?	→ topological plasmonic crystal ?

§ 6 Topological magnonic crystals

Topological chiral modes in magnonic crystals

- R. Shindou, R. Matsumoto, J. Ohe, S. Murakami, Phys. Rev. B 87,174427 (2013)
- R. Shindou, J. Ohe, R. Matsumoto, S. Murakami, E. Saitoh,Phys. Rev. B87, 174402 (2013)

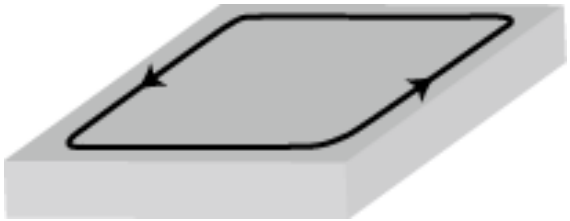


Chern number & topological chiral modes

Band gap → Chern number for n-th band = integer

$$\mathrm{Ch}_n = \grave{o}_{BZ} \frac{d^2k}{2p} W_n\left(\overset{r}{k}\right) \qquad \vec{\Omega}_n(\vec{k}) = i\left\langle \frac{\partial u_n}{\partial \vec{k}} \middle| \times \middle| \frac{\partial u_n}{\partial \vec{k}} \right\rangle$$

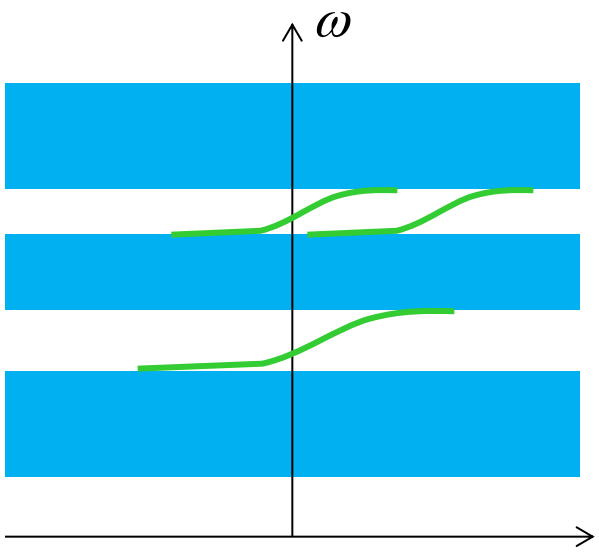
Berry curvature



topological chiral edge modes

$$\mathring{a}_{n\hat{I}\ \text{bands below } E} \mathrm{Ch}_n = N\ ^{o}\ \#(\text{clockwise chiral edge states in the gap at } E)$$

- Analogous to chiral edge states of quantum Hall effect.
- N>0 → cw, N<0: ccw mode



bulk mode: Chern number= Ch_3

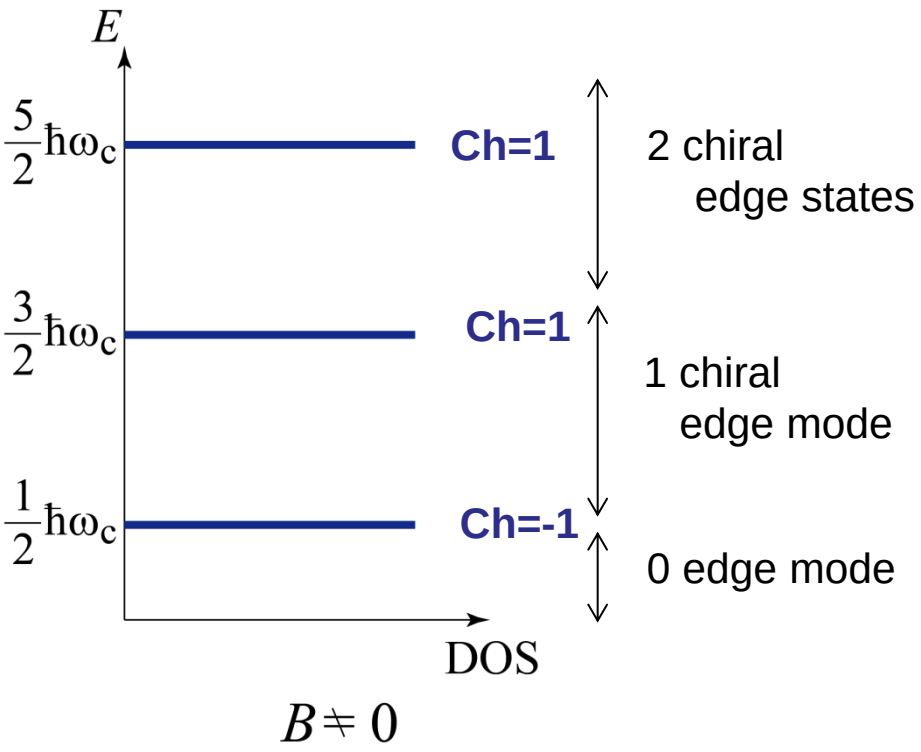
(Ch_1+Ch_2) topological edge modes

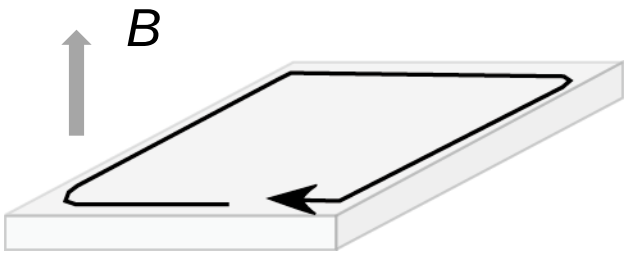
bulk mode: Chern number= Ch_2

Ch_1 topological edge modes

bulk mode: Chern number= Ch_1

Integer quantum Hall effect

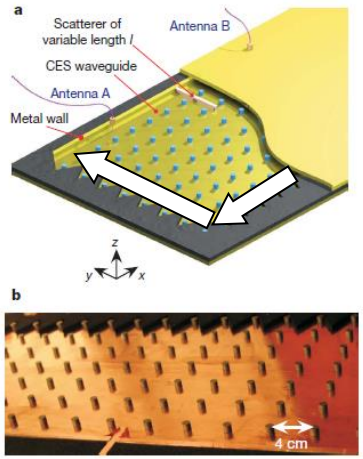




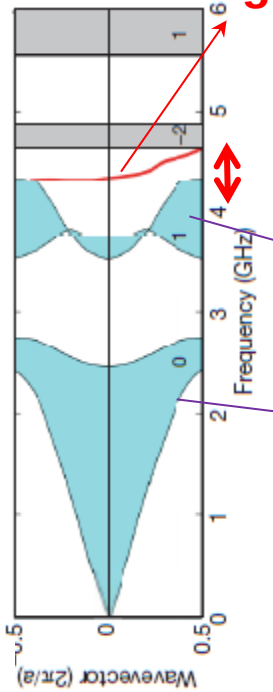
Topological photonic crystals

Theory: Haldane, Raghu, PRL100, 013904 (2008)

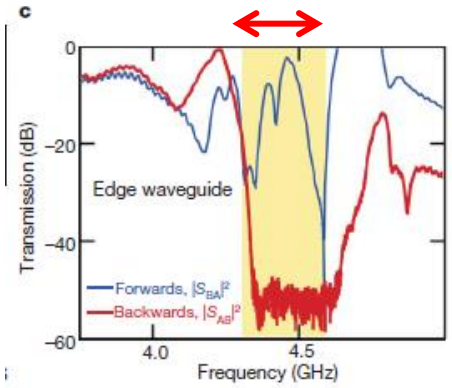
Experiment: Wang et al., Nature 461, 772 (2009)



1 Chiral edge mode



Photonic gap

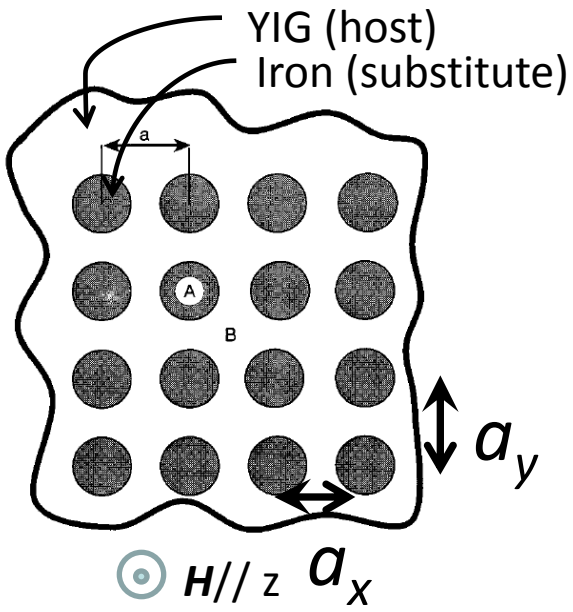


2D Magnonic Crystal : periodically modulated magnetic materials

◆ Landau-Lifshitz equation $\dfrac{\partial \boldsymbol{M}}{\partial t} = -\gamma\, \boldsymbol{M} \times \boldsymbol{H}_{\mathrm{eff}}$

◆ Maxwell equation (magnetostatic approx.)

$$\nabla \times \boldsymbol{H} = \boldsymbol{0},$$
$$\nabla \cdot \left(\boldsymbol{H} + 4\pi \boldsymbol{M}\right) = 0.$$



- Saturation magnetization M_s
- exchange interaction length Q

} modulated

◆ Linearized EOM

$$\frac{1}{|\gamma|\mu_0}\frac{\partial m_\pm}{\partial t} = \mp i H_0 m_\pm \pm 2iM_s(\nabla \cdot Q\nabla)m_\pm \mp 2im_\pm(\nabla \cdot Q\nabla)M_s \pm ih_\pm M_s.$$

External field

exchange field (quantum mechanical short-range)

Dipolar field (classical, long range)

$$m_\pm \equiv m_x \pm i m_y.$$
$$h_\pm \equiv h_x \pm i h_y.$$

$$\Longrightarrow \mathcal{H} = \frac{1}{2}\sum_{\boldsymbol{k}} \left[\, \beta^\dagger_{\boldsymbol{k}} \;\; \beta_{-\boldsymbol{k}} \,\right] \cdot \boldsymbol{H}_{\boldsymbol{k}} \cdot \begin{bmatrix} \beta_{\boldsymbol{k}} \\ \beta^\dagger_{-\boldsymbol{k}} \end{bmatrix}.$$

bosonic Bogoliubov – de Gennes eq.

chiral magnonic band in magnonic crystal

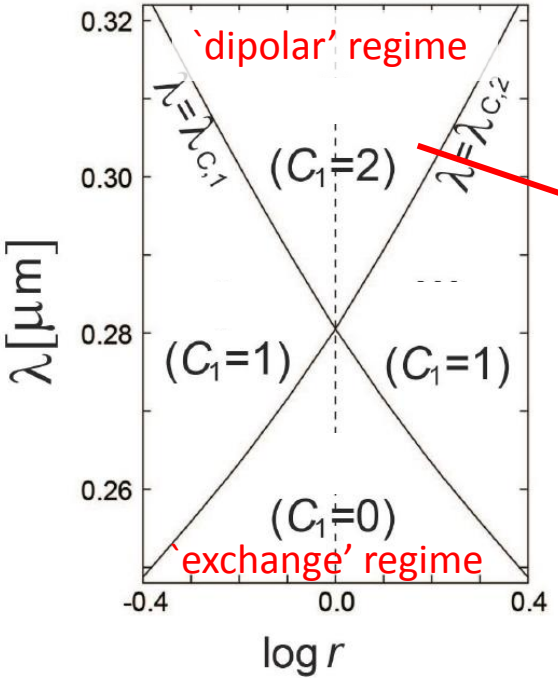
Magnonic gap between 1st and 2nd bands

→C_1 (Chern number) for 1st band

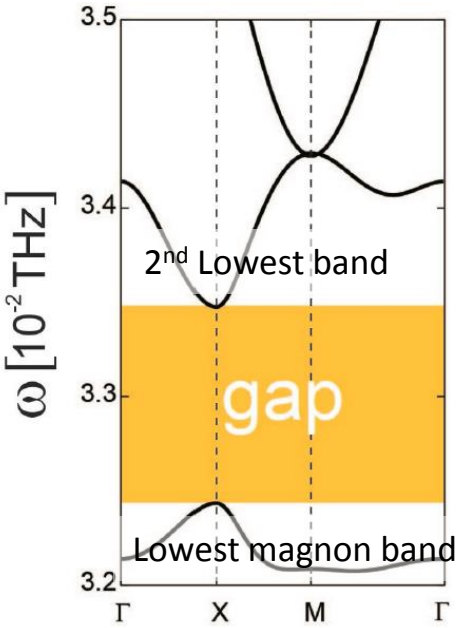
$$C_1 = \grave{O}_{BZ} \frac{d^2k}{2p} W_1(k)$$

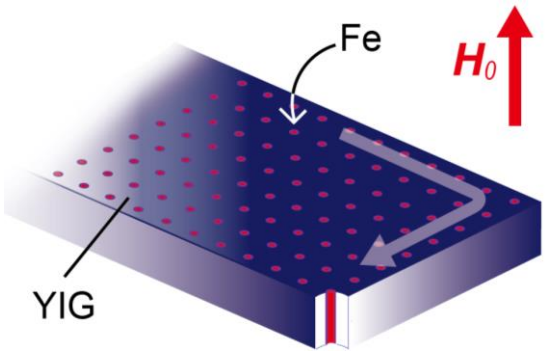
=Number of topological chiral modes within the gap

Phase diagram



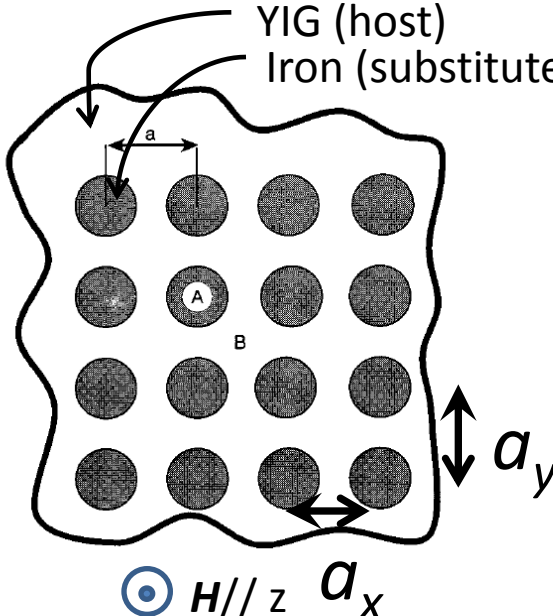
λ=0.35um, r=1





$\lambda = \sqrt{a_x a_y}$: unit cell size

$r = a_y / a_x$: aspect ratio of unit cell

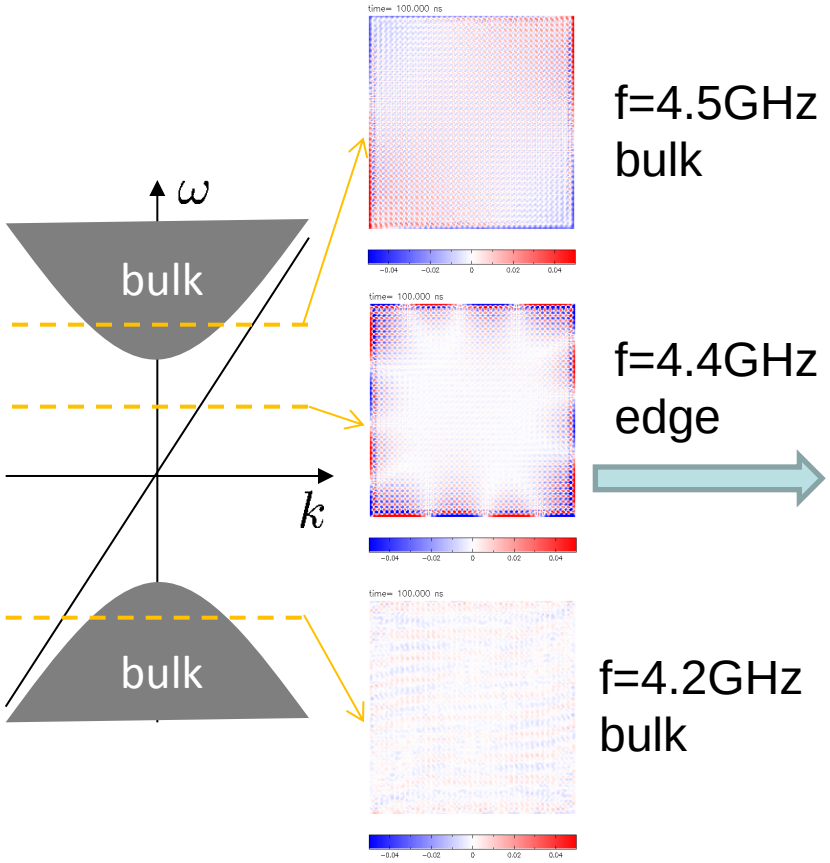


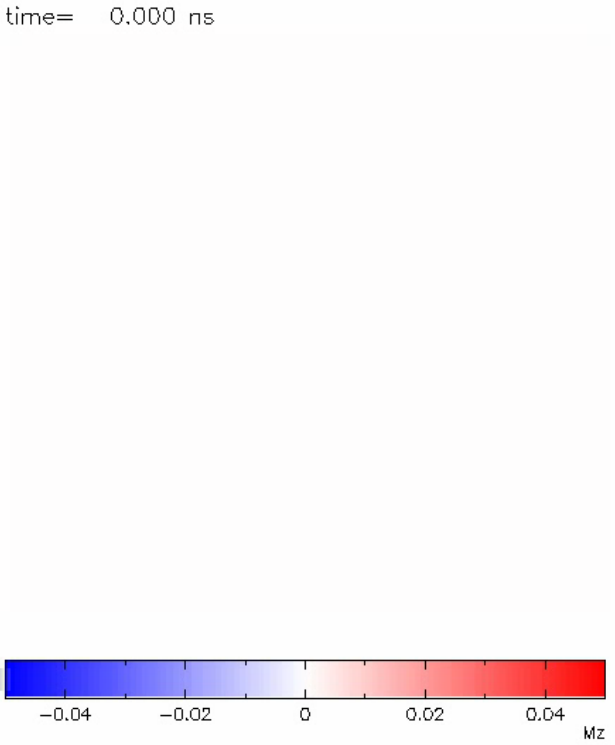
◆ dipolar interaction

➔ non-trivial Chern integer (like spin-orbit interaction)

Simulation (by Dr. Ohe)

DC magnetic field : out-of-plane
AC magnetic field : in-plane





Magnonic crystals with ferromagnetic dot array

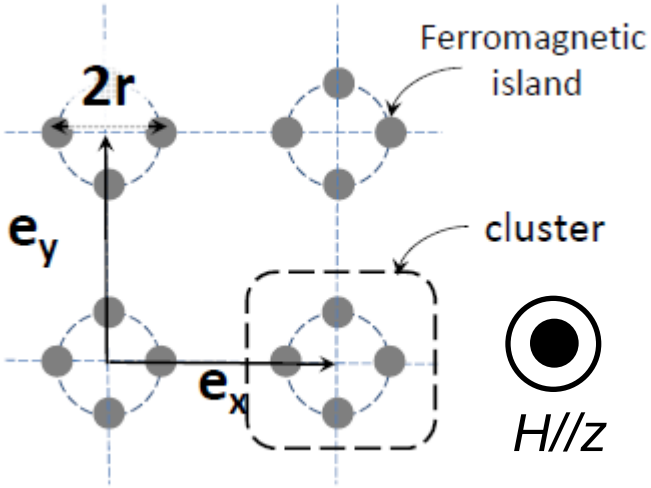
R. Shindou, J. Ohe, R. Matsumoto, S. Murakami, E. Saitoh, PRB (2013)

dot (=thin magnetic disc) → cluster: forming “atomic orbitals”

➔ convenient for (1) understanding how the topological phases appear

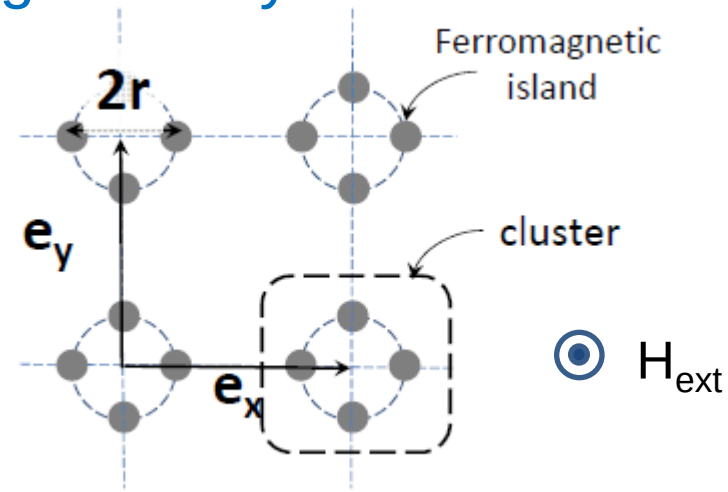
(2) designing topological phases

decorated square lattice



Each island is assumed to behave as monodomain

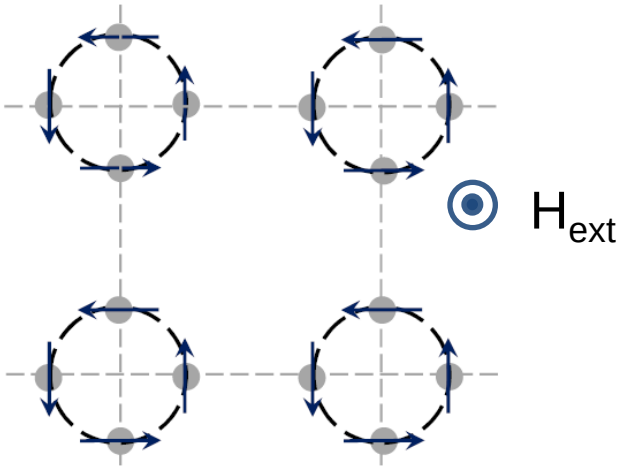
Magnonic crystals: decorated square lattice



$$\left|\vec{e}_x\right| = \left|\vec{e}_y\right| = 2.4,\ 2r = 1.2,\ \mathrm{D}V = 1.70,\ M_s = 1.0$$

Equilibrium spin configuration

$H_{ext} < H_c = 1.71$



Tilted along H_{ext}

Magnetostatic energy

$$E = -\frac{1}{2}(\Delta V)^2 \sum_{i,j}^{i\neq j} M_a(r_i) f_{ab}(r_i - r_j) M_b(r_j) + H\Delta V \sum M_z(r_j).$$

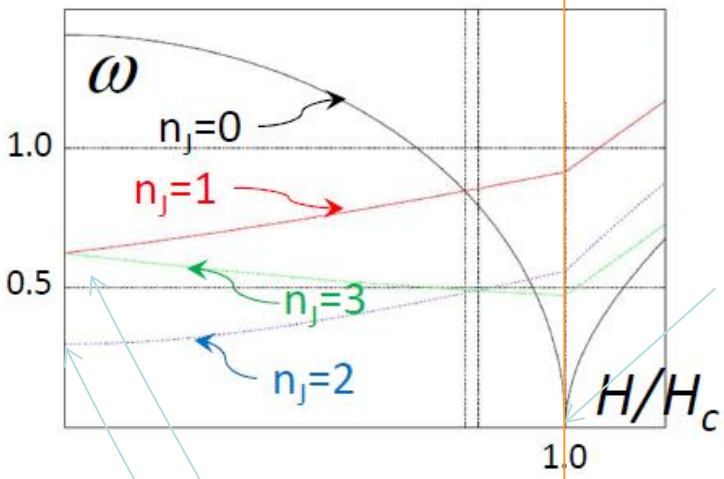
$$f_{ab}(r) = \frac{1}{4\pi}\left(\frac{\delta_{a,b}}{|r|^3} - \frac{3r_a r_b}{|r|^5}\right)$$

$H_{ext} > H_c$

Collinear // H_{ext}

“atomic orbitals” within a single cluster

Energy levels of atomic orbitals

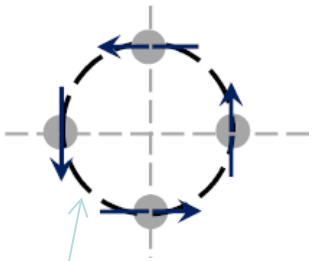


n_J=0 softens at $H=H_c$

n_J=1 and n_J=3 degenerate at H=0

n_J=2 is lowest at H=0: favorable for dipolar int.

Equilibrium configuration

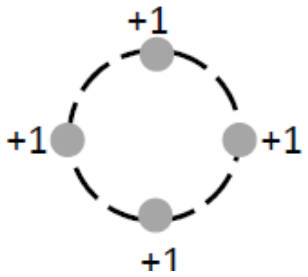


H<H_c: noncollinear

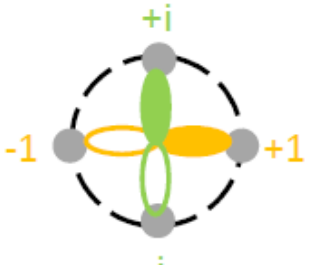
H>H_c: collinear // z

Spin wave excitations: “atomic orbitals”

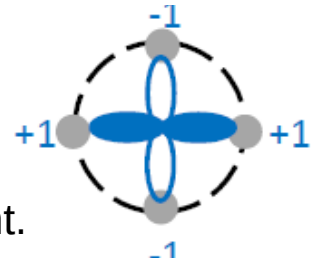
relative phase for precessions



n_J=0 (s-orbital)



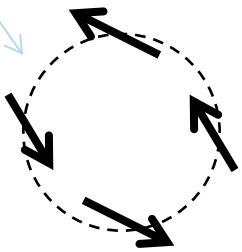
n_J=+1 (p_x+ip_y-orbital)



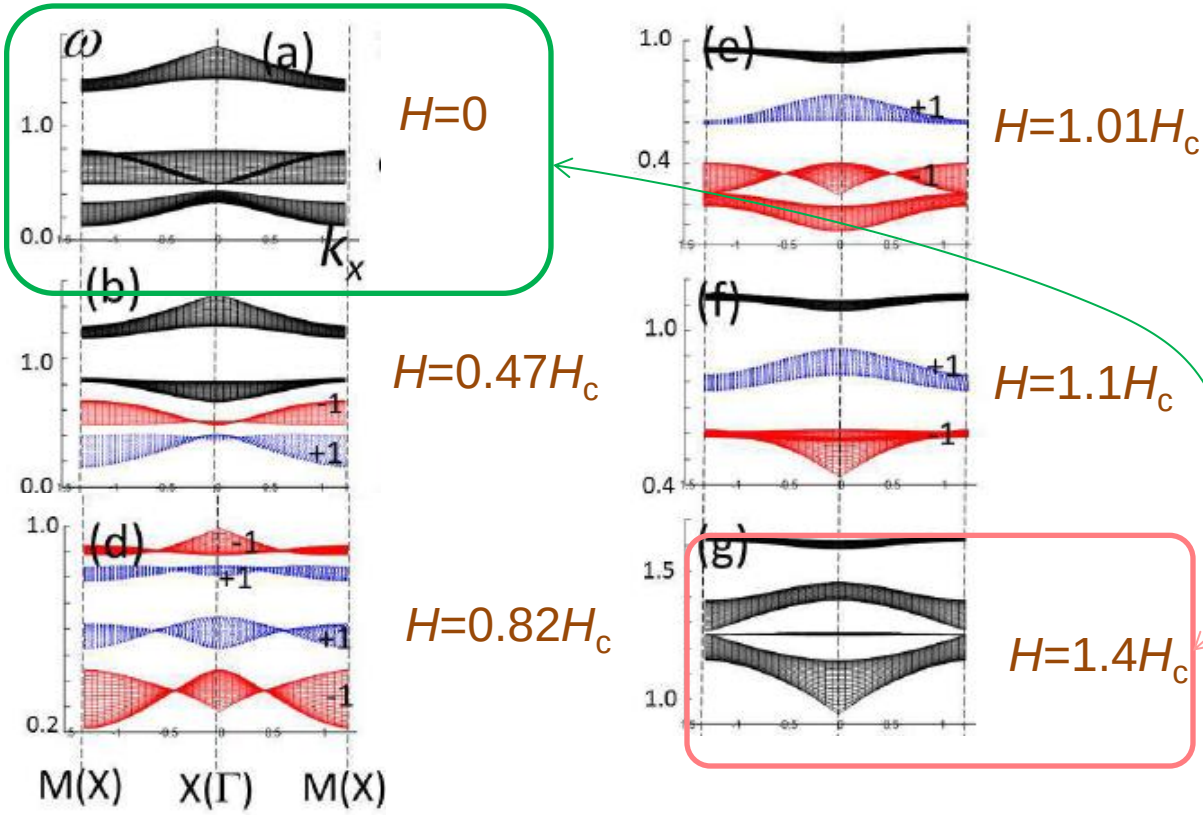
n_J=2 (dx^2-y^2-orbital)



n_J=3 (p_x-ip_y-orbital)



- Spin-wave bands and Chern numbers



Red: Ch=-1
Blue: Ch=+1
→Topologically nontrivial = chiral edge modes

Time-reversal symmetry

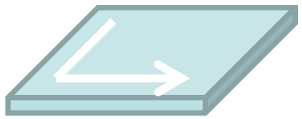
- Small $H \ll H_c$ →Topologically trivial
- Large $H \gg H_c$

Dipolar interaction is weak

→Nontrivial phases (i.e. nonzero Chern number) in the intermediate magnetic field strength

topological chiral edge modes

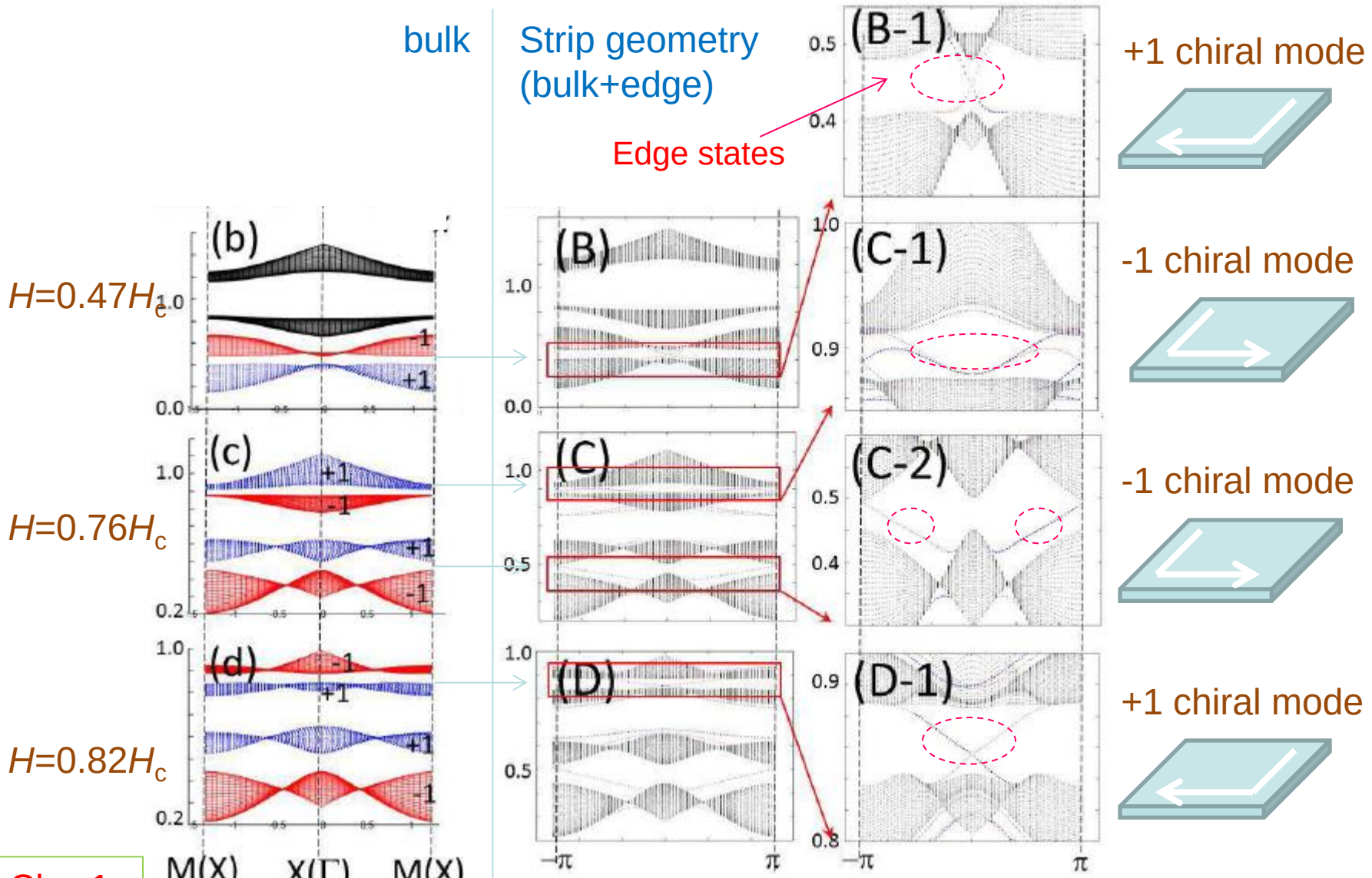
-1 chiral mode



+1 chiral mode



Magnonic crystals: edge states and Chern numbers (1)



Magnonic crystals: tight-binding model with atomic orbitals

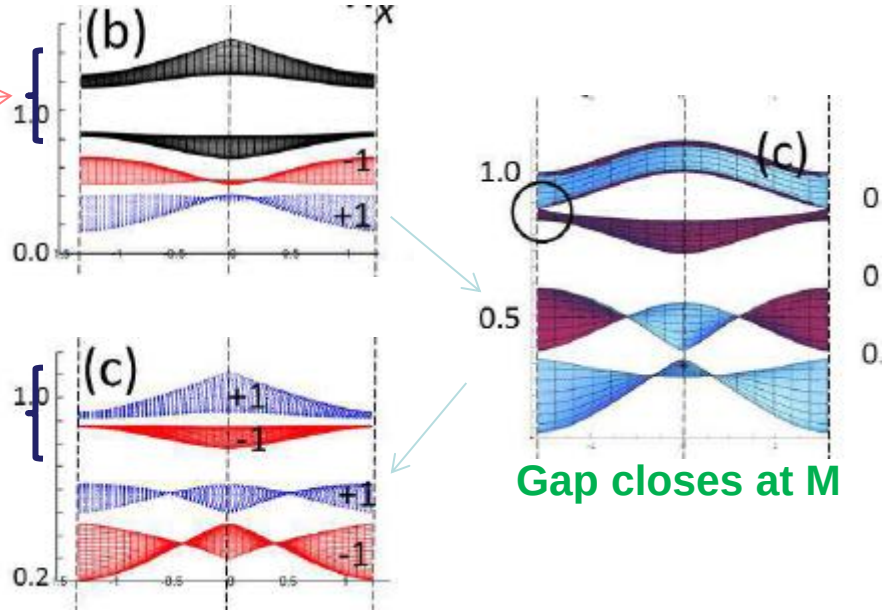
(example) :

H=0.47H_c → H=0.82H_c

gap between 3rd and 4th bands

retain only n_J=0 and n_J=1 orbitals

→ tight binding model



$$H_{01,\boldsymbol{k}} = \begin{pmatrix} \varepsilon_0 + 2a_{00}(c_{k_x} + c_{k_y}) & -2ib_{01}(s_{k_x} - is_{k_y}) \\ 2ib_{01}(s_{k_x} + is_{k_y}) & \varepsilon_1 + 2a_{11}(c_{k_x} + c_{k_y}) \end{pmatrix}$$

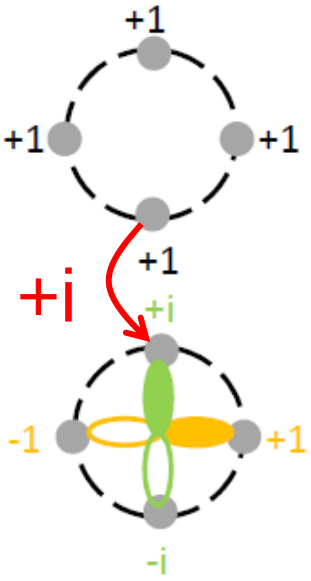
complex phase for hopping

← p_x+ip_y orbitals

= Model for quantum anomalous Hall effect

(e.g. Bernevig et al., Science 314, 1757 (2006))

→Gap closing + topological transition

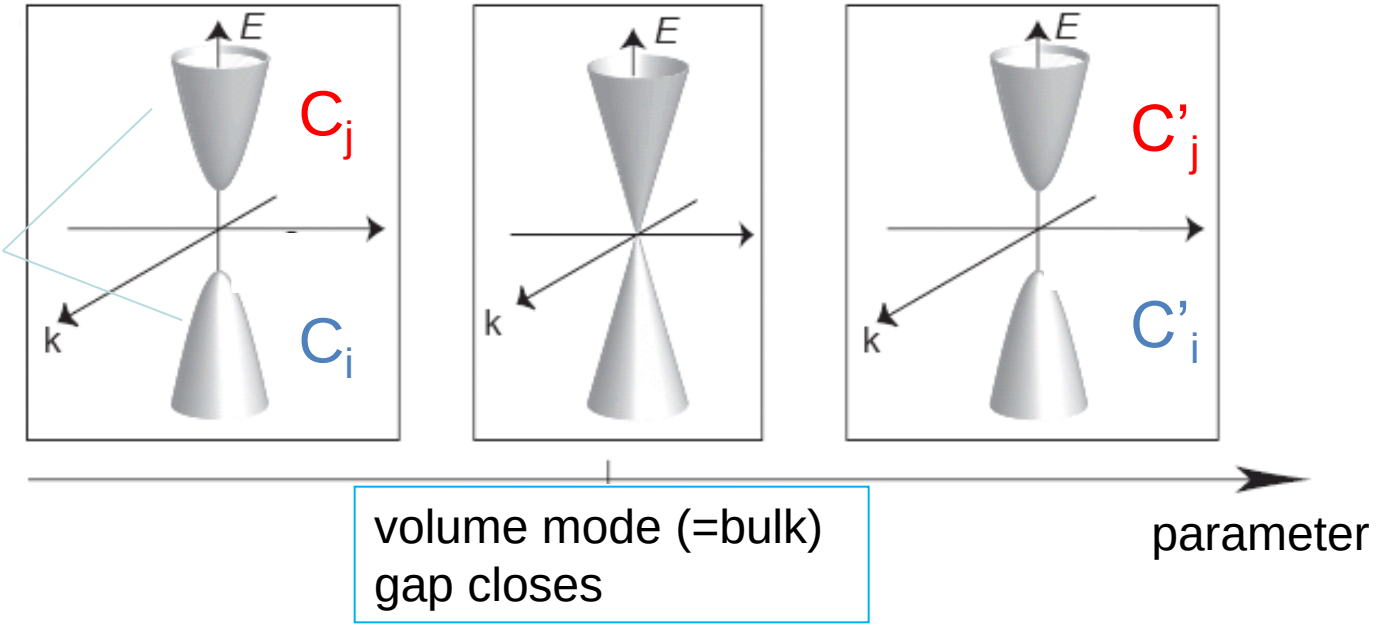


n_J=0
(s-orbital)

n_J=+1
(p_x+ip_y-orbital)

Change of Chern number at gap closing event

magnonic
(volume-mode)
bands



$\Delta C_i = -\Delta C_j = \pm 1$

: Dirac cone at gap closing

($\rightarrow C_i + C_l = C'_i + C'_l$: sum is conserved)