Merging of Exceptional Points in Classical Waves: Acoustics and photonic crystals

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Outline

- Background and motivations

- Part I. Acoustics

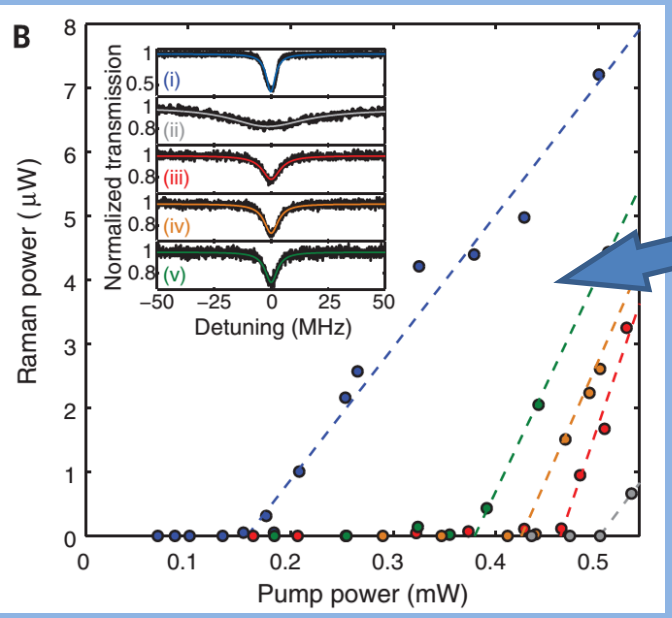
1. Two coupled cavity resonators
2. Four-state non-Hermitian Hamiltonian, and eigen-frequency phase diagram
3. Realization of exceptional point formation patterns
4. Topological characteristics around singularities

- Part II. Photonic Crystal (PC)

1. Formation of model Hamiltonian
2. Complex band calculations
3. Two types of phase diagram

- Conclusions

Background: Interesting phenomena

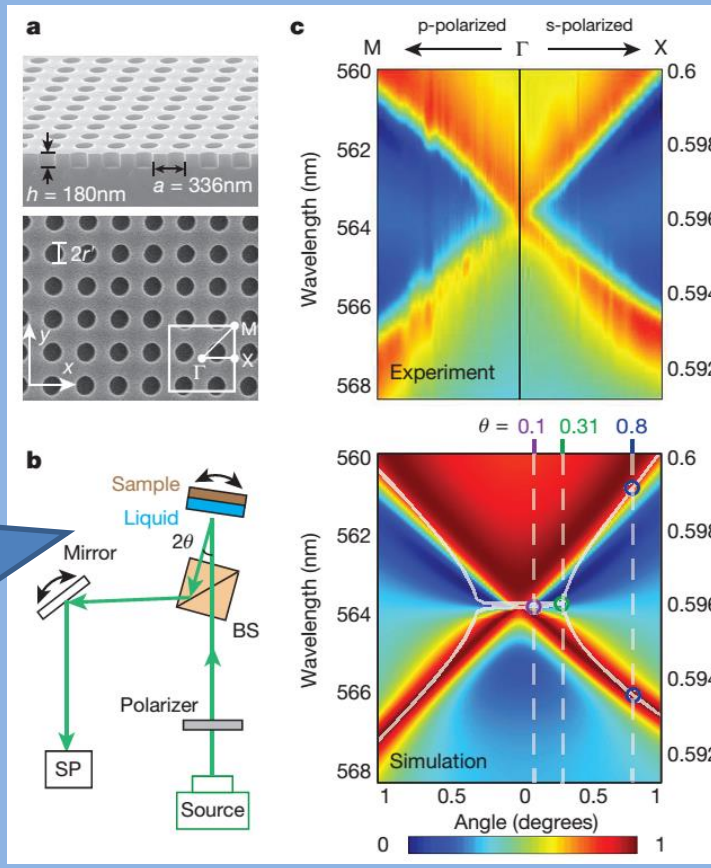


Science **346**, 328 (2014).

Exceptional point

- Loss-induced revival of lasing
- Single mode laser
- Unidirectional reflection
- Coherent perfect absorption
- A ring of exceptional points
- others

Nature **525**, 354 (2015).



Science **346**, 328 (2014); Nat. mat. **12**, 108 (2012); PRL **112**, 143903 (2014); Science **346**, 972 (2014); Science **346**, 975 (2014); Nature **525**, 354 (2015); Nature **526**, 554 (2015); Phys. Rev. A **91**, 053825 (2015).

Background: Two state problem

Model Hamiltonian is

$$\mathrm{H} = \begin{pmatrix} \omega_0 - i\Gamma_0 & \kappa \\ \kappa & \omega_0 - i\Gamma \end{pmatrix}$$

Rotate H to $\Gamma_0 = \Gamma = 0$ representation

$$\mathrm{H} = \begin{pmatrix} \omega_0 - |\kappa| - i\Gamma_{av} & -i\Delta\Gamma/2 \\ -i\Delta\Gamma/2 & \omega_0 + |\kappa| - i\Gamma_{av} \end{pmatrix}$$

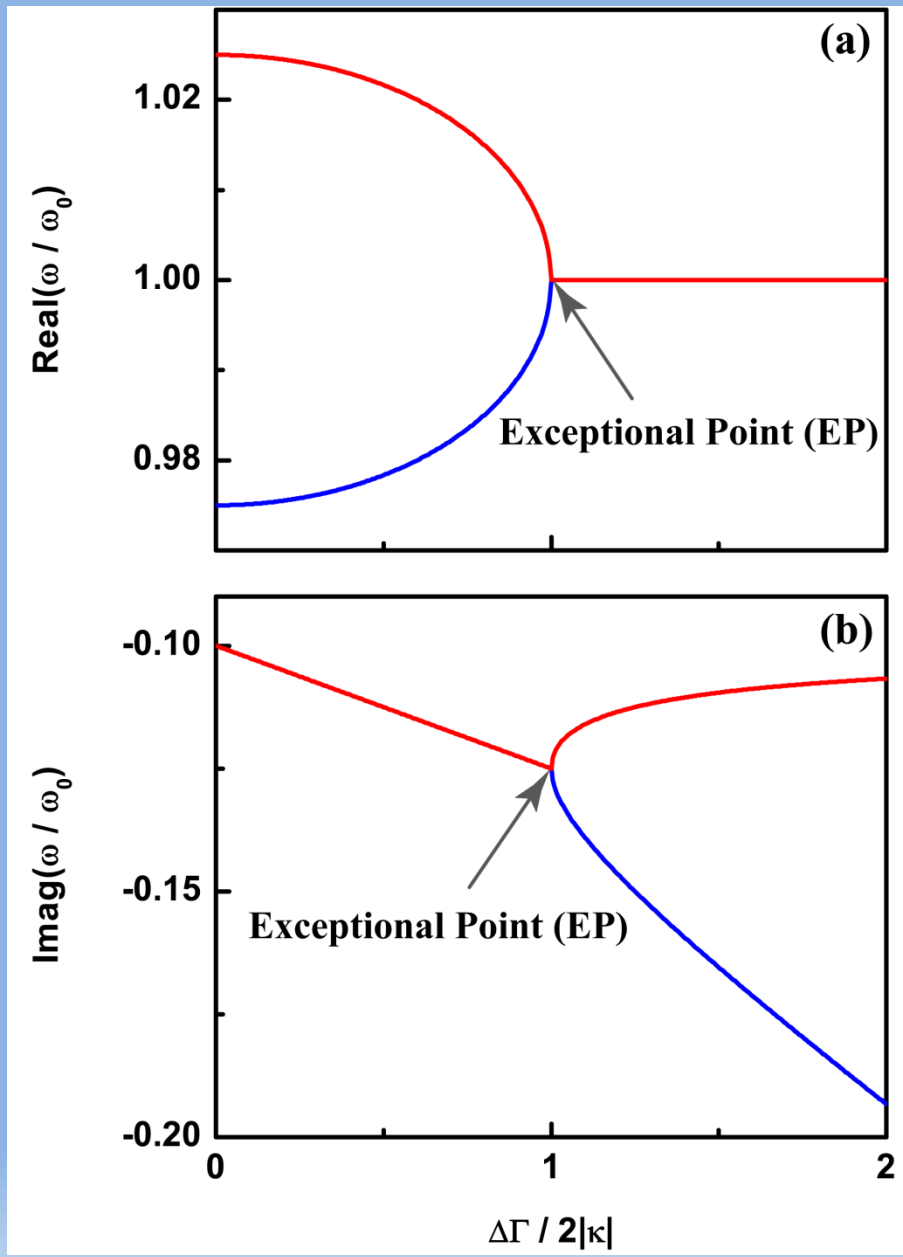
where $\Gamma_{av} = (\Gamma_0 + \Gamma)/2$ and $\Delta\Gamma = \Gamma - \Gamma_0$.

This means **loss difference** $\Delta\Gamma$ plays the role of **"attractive potential"**.

Eigenvalues are

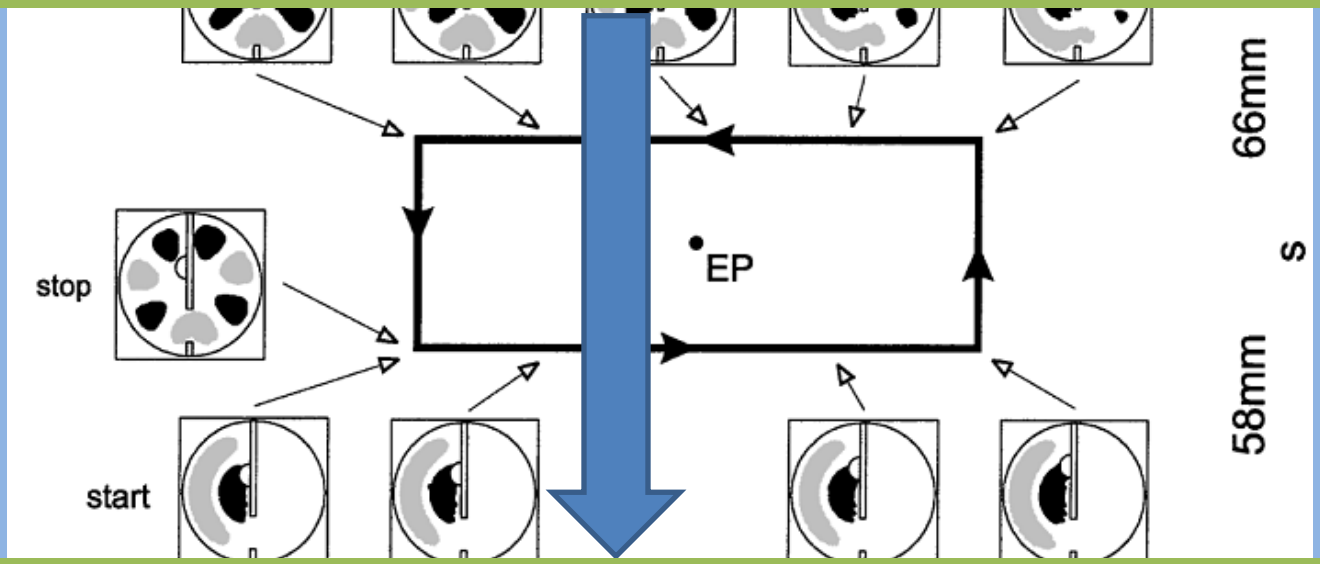
$$\tilde{\omega}_{\pm} = \omega_0 - i\Gamma_{av} \pm \frac{1}{2}\sqrt{4\kappa^2 - (\Delta\Gamma)^2}$$

- Exceptional Point (EP) is formed at $(\Delta\Gamma)^2 = 4\kappa^2$;
- Real parts of eigen-frequencies are separated before EP, and become identical after EP.



Background: singularity orders

All EP-related phenomena comes from square-root singularity, and can be described by 2x2 matrix locally.

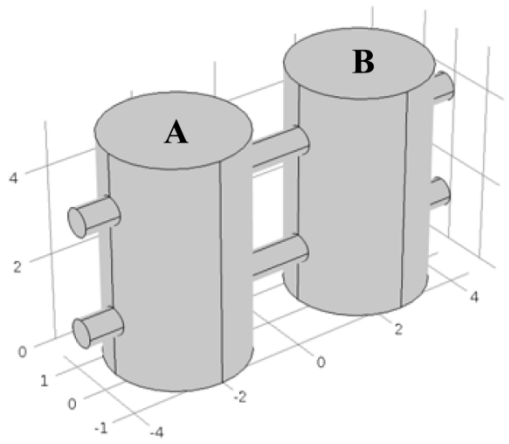


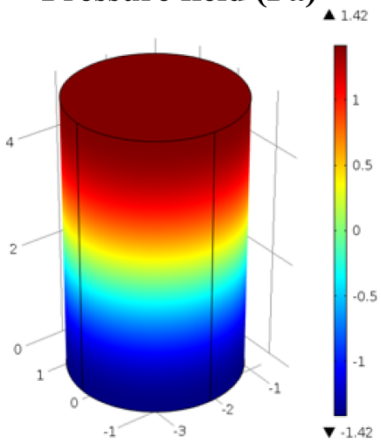
Motivations:

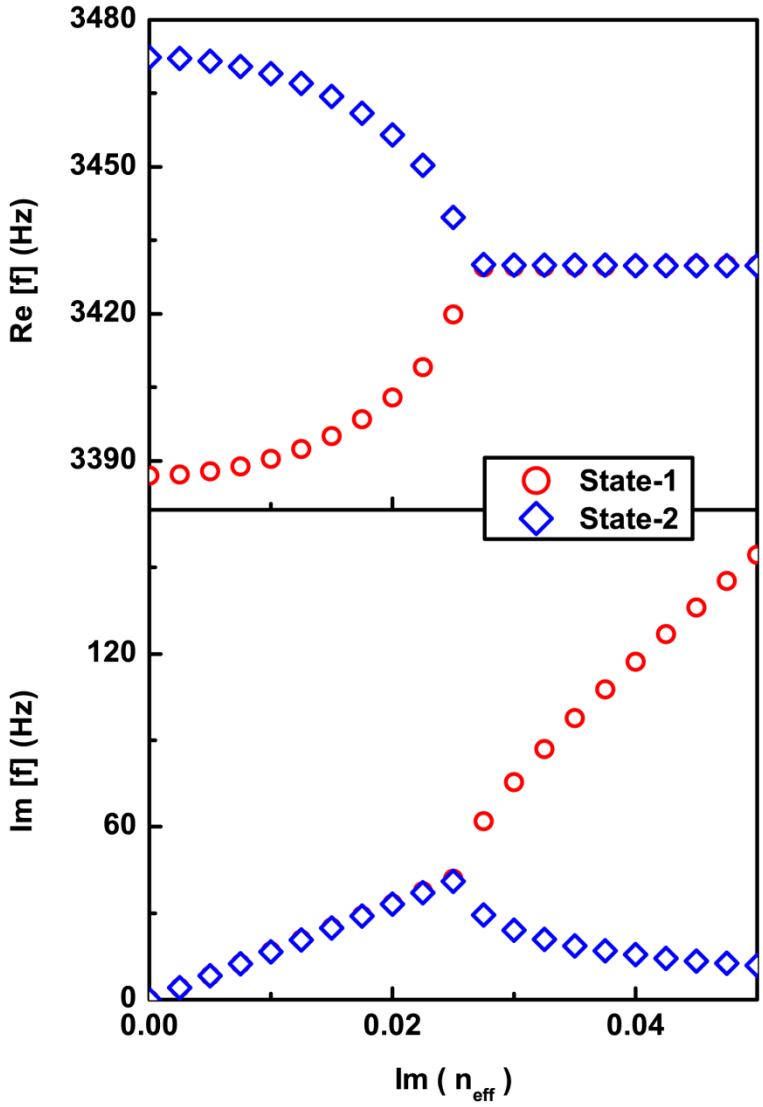
- What will happen for the system which can NOT reduce to 2x2?
- Does the interaction of multiple exceptional points lead to new singularities and new topological characteristics that are qualitatively different from those in 2x2?

- Experimental Observation of the Topological Structure of Exceptional Points, Phys. Rev. Lett. **86**, 787 (2001).
- Optical realizations: Nature **526**, 554 (2015).

A I. Design of acoustic Hamiltonian



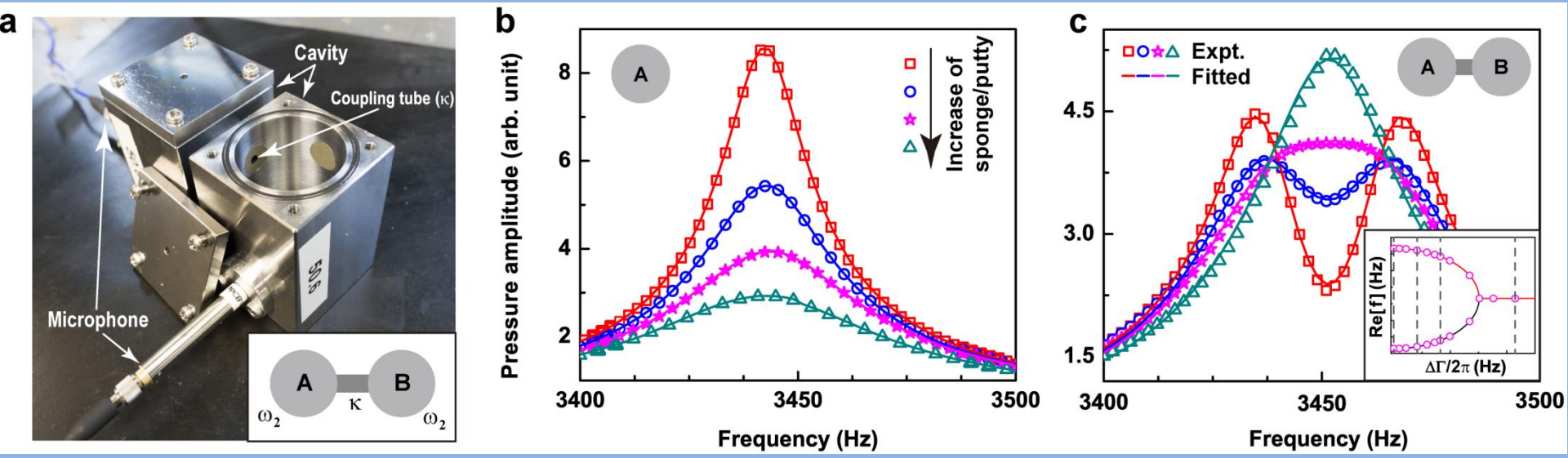




- Local resonance are achieved by the resonance mode in z-direction with frequency $\omega = \pi v / h$ where v is speed of sound, and h is height of the cavity.
- Additional loss was introduced at cavity B.
- Simulation results show the existence of exceptional point (EP).

Height of A and B is 50.0mm, and radius of A and B is 15.0mm. Radius and length of connecting tubes between A and B are 3mm and 15mm, respectively.

Experimental realizations

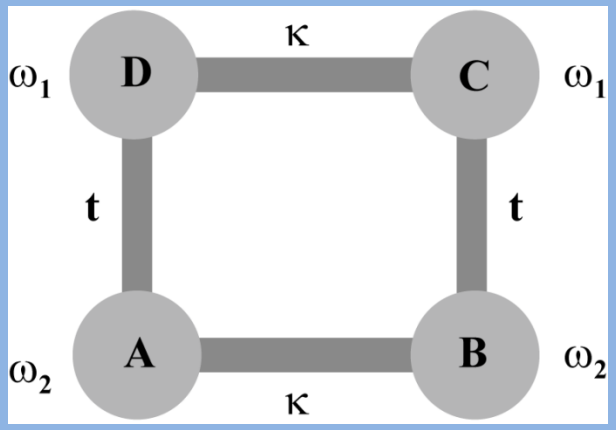


- The height of Cavities A and B have the same height 50.6mm, and radii of the coupling tubes are 2.0mm and 0.8mm, respectively. Asymmetric loss is introduced in cavity B by adding a mixture of sponge and putty into the cavity, whereas both pumping and measurements are at cavity A.
- As shown in **b**, the introduce of sponge and putty only increase loss to the cavity;
- Response of given source $|s\rangle$ and probe $|p\rangle$ is $R(\omega)=\langle p|\vec{\vec{G}}(\omega)|s\rangle$, where dyadic Green's function by using bilinear product is

$$\vec{\vec{G}}(\omega)=\sum_{i=1}^{N}\frac{|\phi_i^R\rangle\langle\phi_i^L|}{\omega-\omega_i}$$

- As shown in **c**, two peaks merge into one peak with increasing loss.

A II. Four-state non-Hermitian Hamiltonian



$$H=\begin{pmatrix} \omega_2-i\Gamma_0 & \kappa & & t \\ \kappa & \omega_2-i\Gamma & t & \\ & t & \omega_1-i\Gamma_0 & \kappa \\ t & & \kappa & \omega_1-i\Gamma \end{pmatrix},$$

in which AB and CD are two pairs of resonators, $\omega_0=(\omega_1+\omega_2)/2$, $\Delta\omega=\omega_1-\omega_2$, Γ_0 denotes the intrinsic loss of each cavity, and $\Gamma=\Gamma_0+\Delta\Gamma$ with $\Delta\Gamma$ being a tunable loss at B/D only. Eigenfrequencies of Hamiltonian are

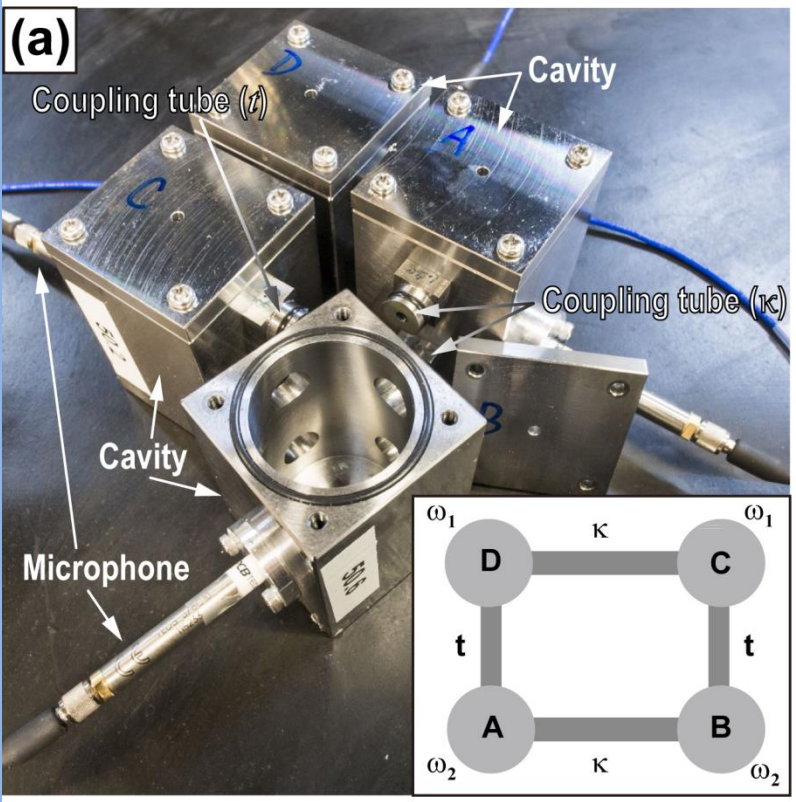
$$\tilde{\omega}_j=\omega_0-i\frac{\Gamma+\Gamma_0}{2}\pm\frac{1}{2}\sqrt{\Delta_1\pm4\sqrt{\Delta_2}}\,,\qquad j=1,2,3,4$$

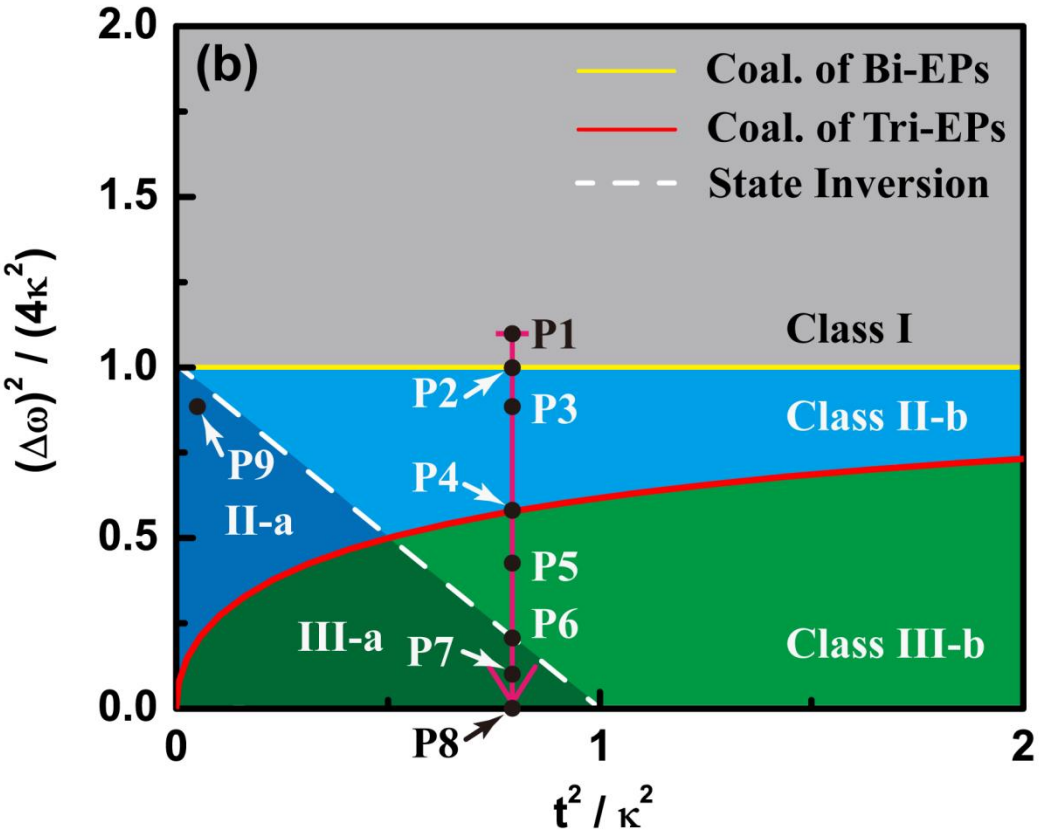
where $\Delta_1=-(\Delta\Gamma)^2+4\kappa^2+4t^2+(\Delta\omega)^2$, and $\Delta_2=4\kappa^2t^2+\kappa^2(\Delta\omega)^2-(\Delta\Gamma)^2\frac{(\Delta\omega)^2}{4}$.

Three cases when coalescence of states (CS) could occur:

- CS-$1_{\pm}$: $\Delta_1\pm4\sqrt{\Delta_2}=0$, one state become defective;
- CS-2: $\Delta_2=0,\ \Delta_1\neq0$, two states become defective;
- CS-3: $\Delta_1=\Delta_2=0$, three states become defective;

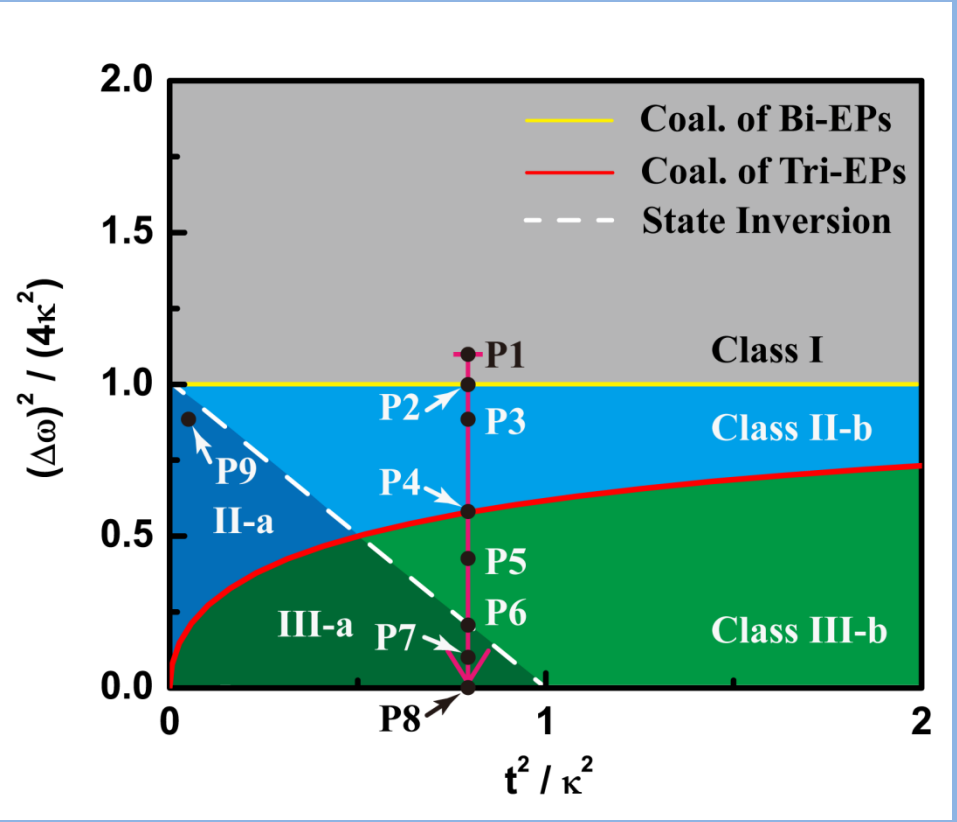
A III. Phase diagram, experiment realizations





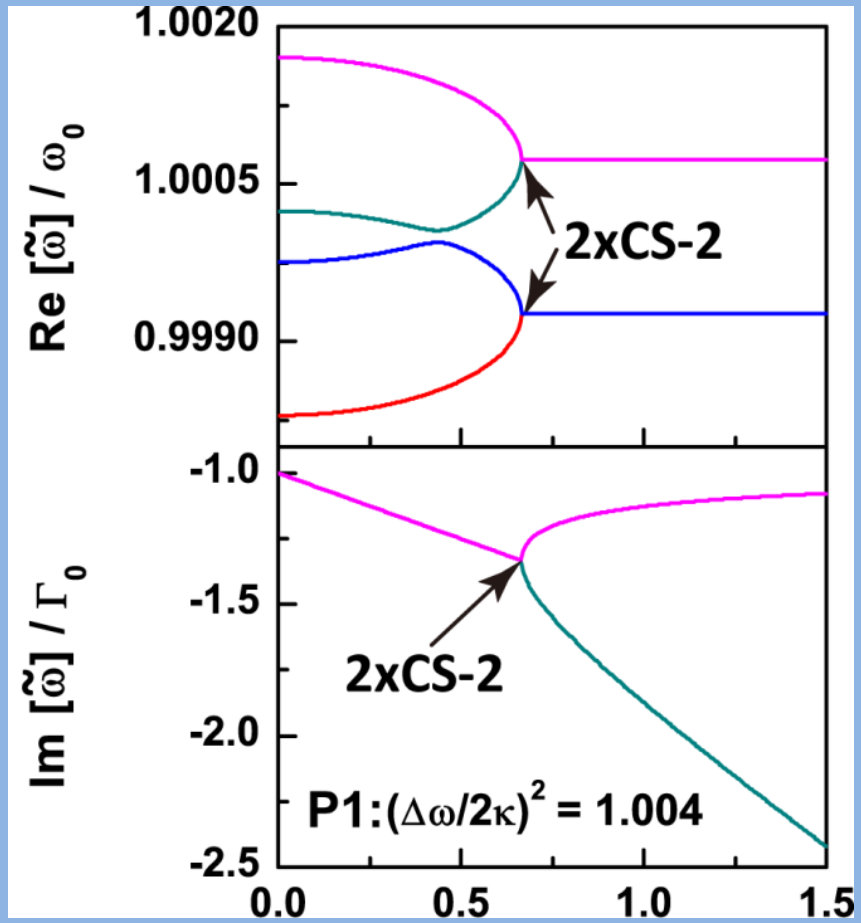
- Four pairs of cylindrical metallic cavities with height 50.0mm, 50.2mm, 50.4 mm and 50.6mm respectively, are made, and radius of them are all 15mm;
- The cavities are filled with air at one atmospheric pressure, with temperature kept at 295 K.
- Measurements are performed four times with the loudspeaker driving each cavity individually, which means incoherent pumping.

1. Class I



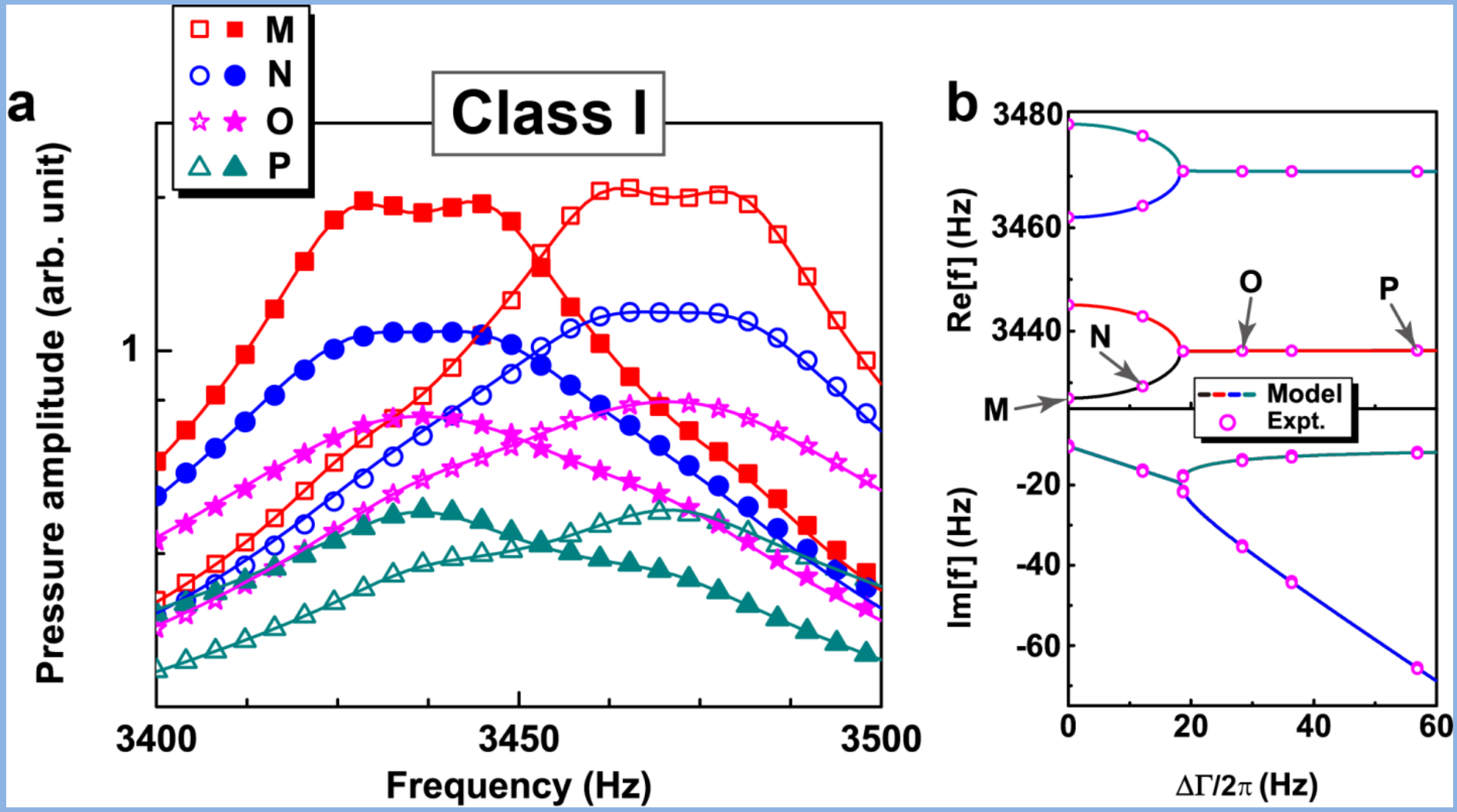
Three special lines and classes exist in the phase diagram:

- Solid red line: CS-3 singularities;
- Solid yellow line: Coalescence of Bi-EPs;
- Dash white line: State inversion;
- Each class has a distinct EP formation pattern (EPFP).



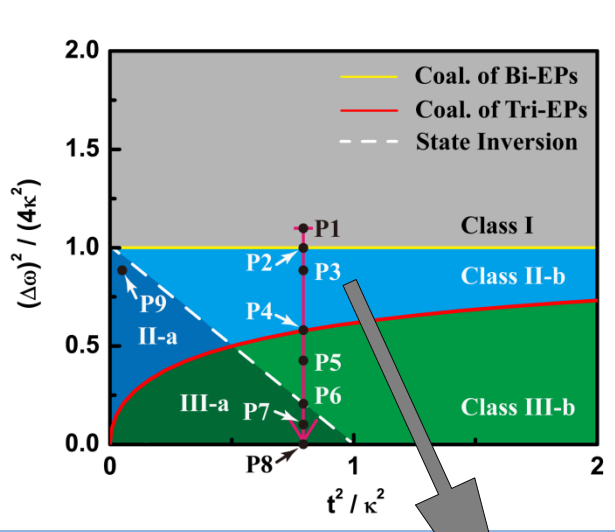
- The inter-pair frequency difference is large enough, so there are still two CS-2 singularity in the spectrum.
- This is a typical EPFP of **Class I**.

Experimental realization of Class I

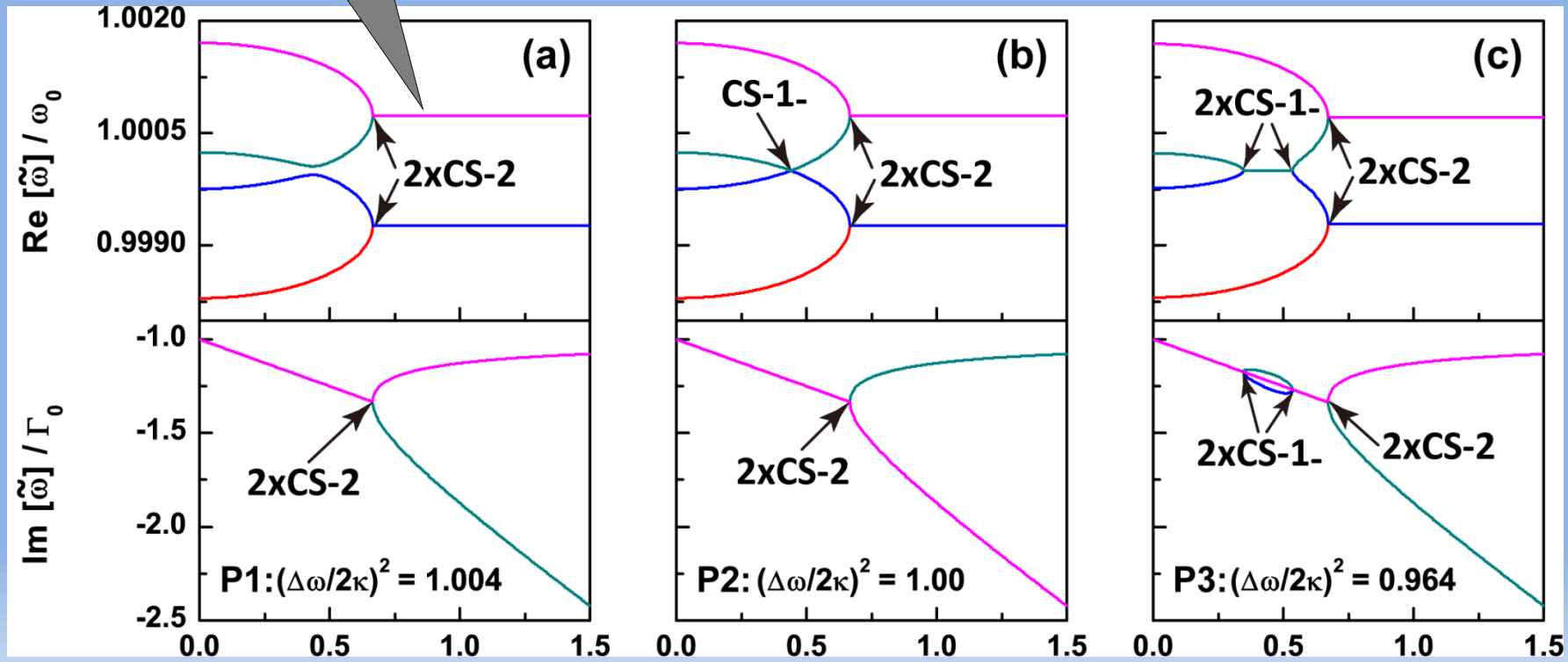


- The height of A/B is 50.6mm with 150mg putty, the height of C/ D is 50.0mm.
- Two κ-tubes, with the same radius of 1.2mm, connect A with B (and C with D).
- Two *t*-tubes, with 0.8mm and 0.4mm in radius, connect A with D (and B with C).
- Filled dots (open dots) are measured at cavity B (D).

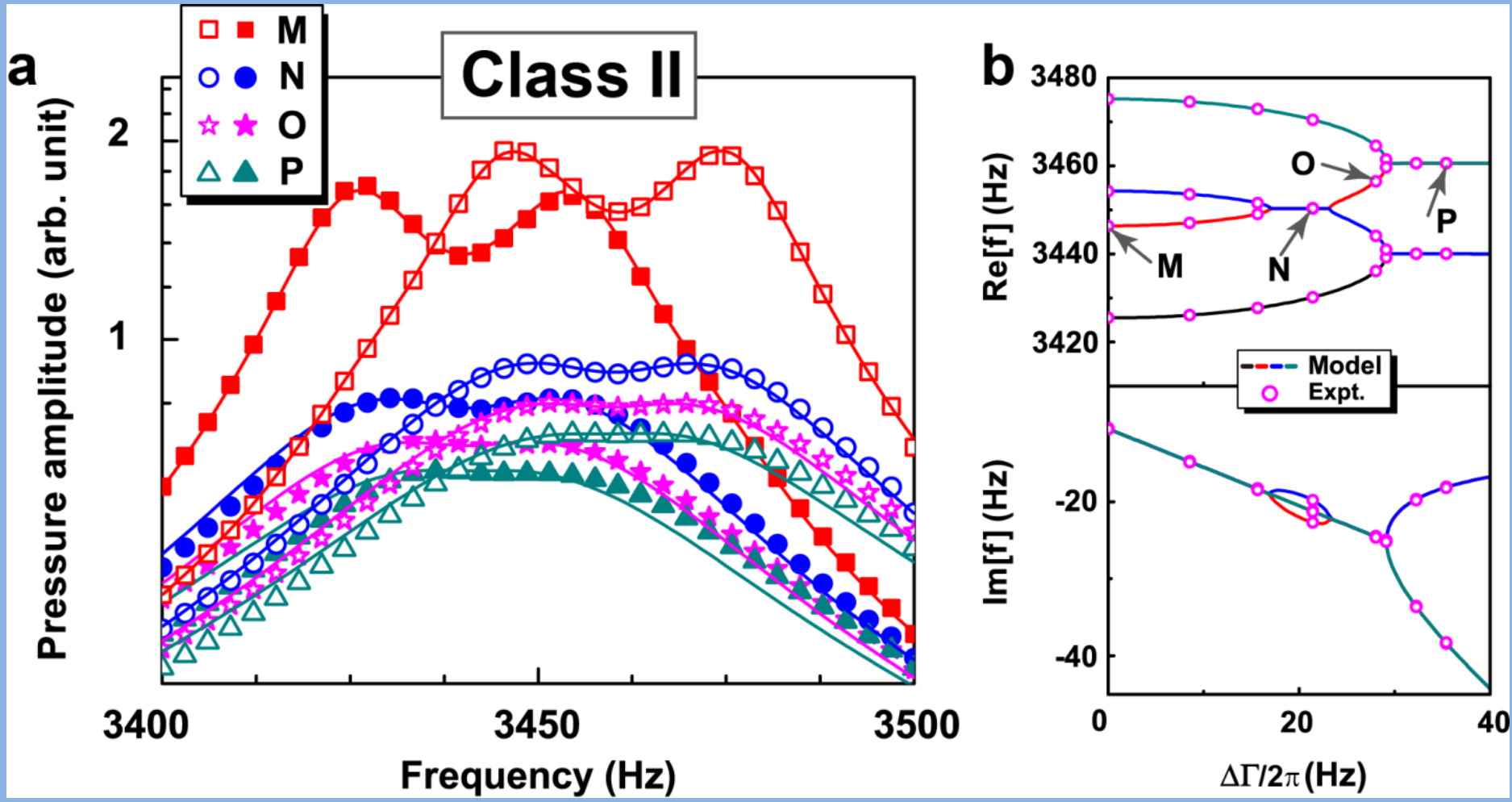
2. Class II and coalescence of two EPs



- Decreasing $\Delta\omega$ will make the gap close, and create a linear crossing (P1 to P2);
- The linear crossing point will split into two CS-1_ EPs (P2 to P3);
- Detailed analysis around the crossing point will given later;
- The spectrum of P3 represents a typical EPFP of **Class II**.

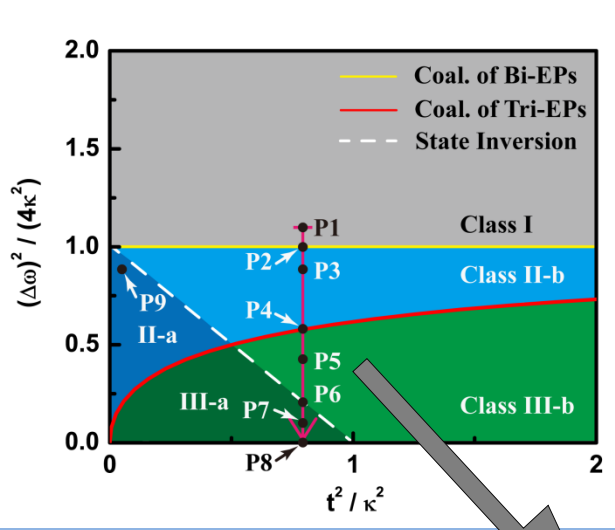


Experimental realization of Class II



- The height of A and B is 50.6mm, and the height of C and D is 50.2mm.
- Two κ-tubes, with 2.0mm and 0.8mm in radius, connect A with B (and C with D).
- Two *t*-tubes with the same radius of 0.4mm connect A with D (and B with C).
- Filled dots (open dots) are measured at cavity B (D).

3. Class III and coalescence of three EPs

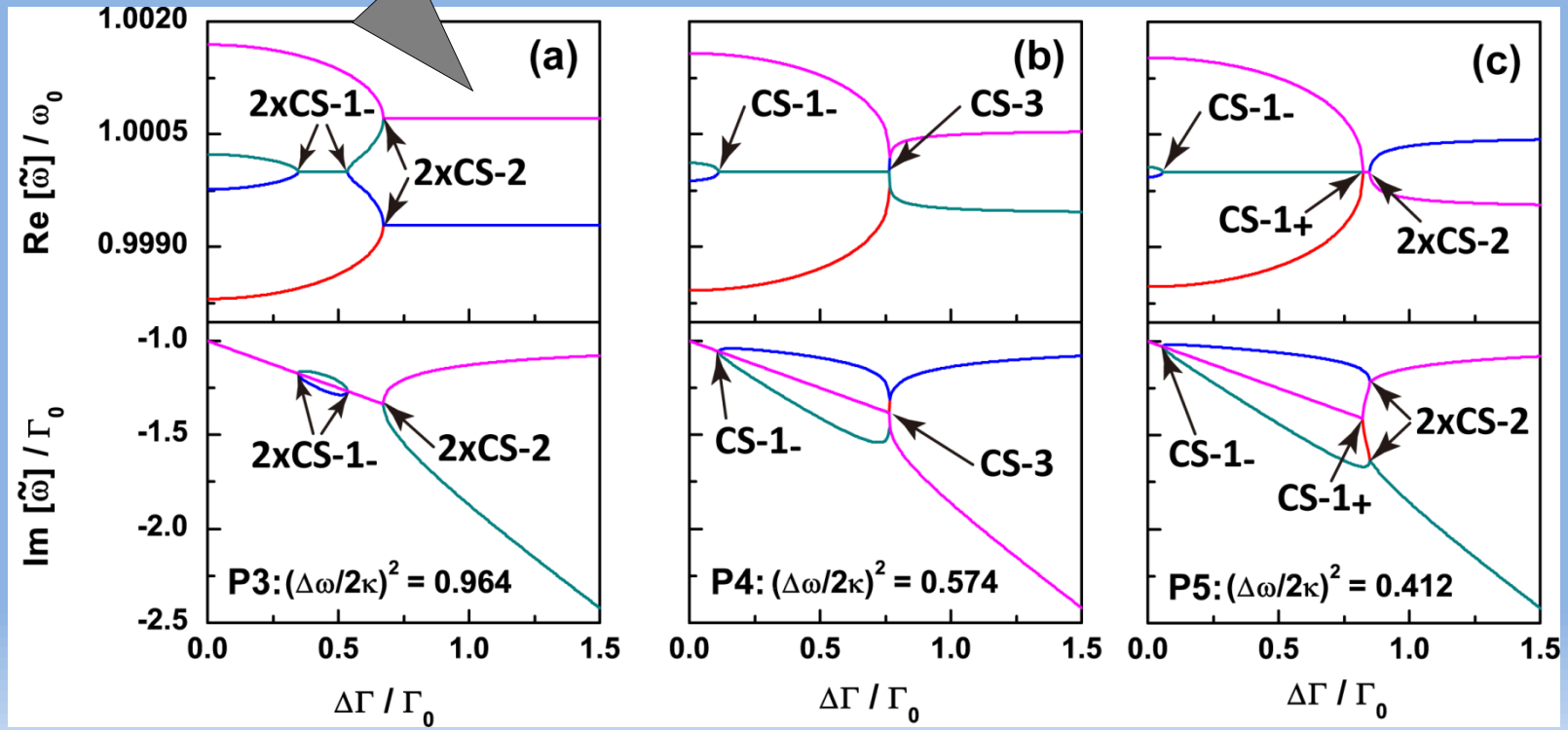


- CS-3 singularity occurs at

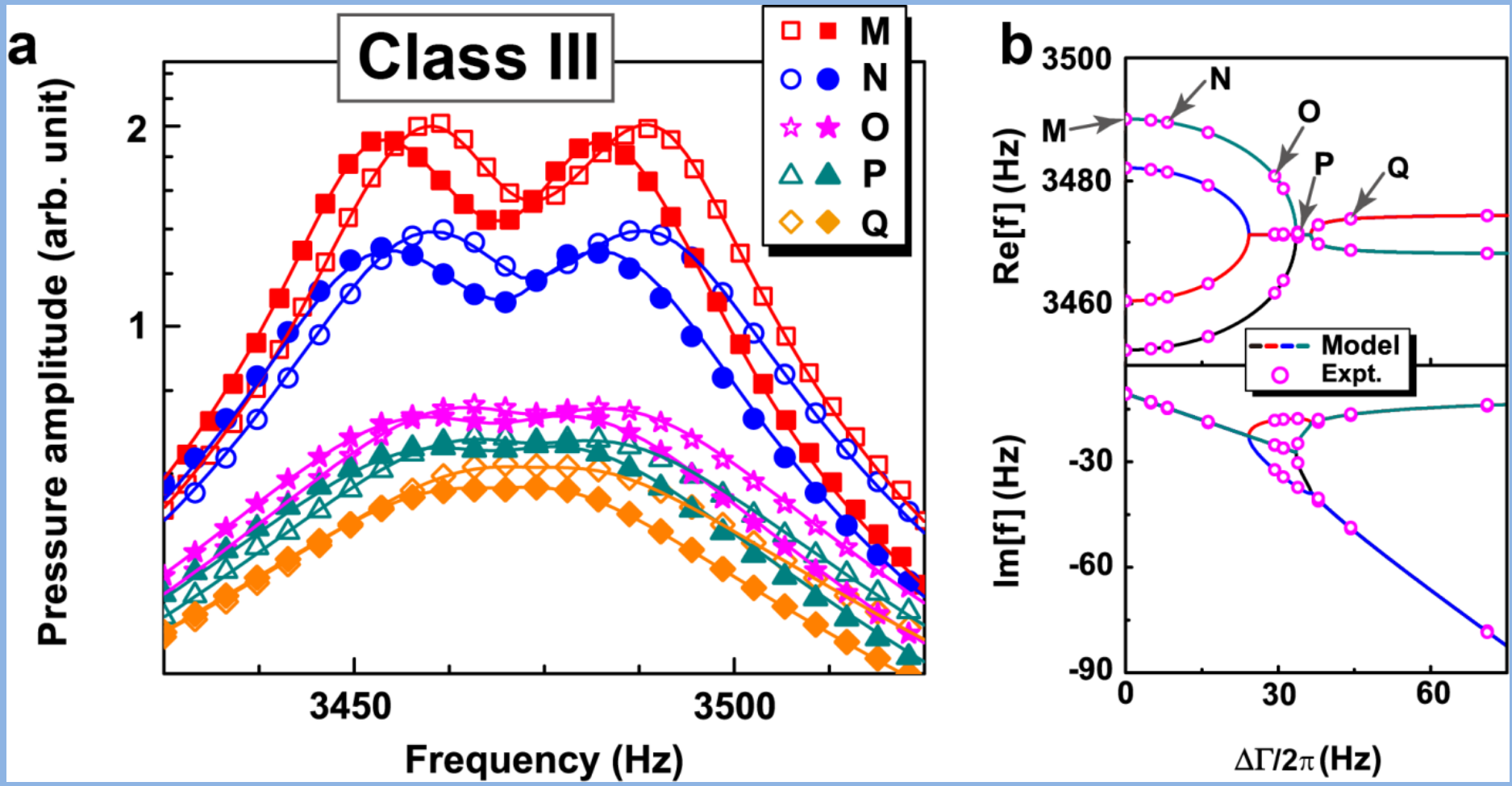
$$(\Delta\omega)^2_{ks} = 2\sqrt{t^4 + 4\kappa^2 t^2} - 2t^2$$

$$\Gamma^2_{ks} = 2|t|\sqrt{t^2 + 4\kappa^2} + 4\kappa^2 + 2t^2$$

- We also call this point as the **kissing** point because CS-1 and 2xCS-2 bifurcations kiss (P4).
- Fig. (c) represents a typical EPFP of **Class III**.

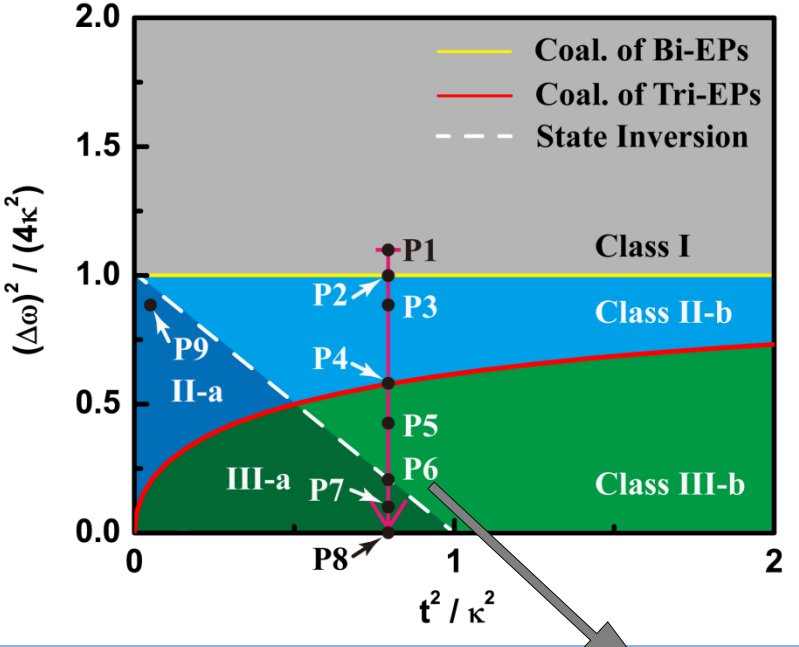


Experimental realization of Class III



- The height of A/B is 50.2mm with 150mg putty, and the height of C/D is 50.0mm.
- Two κ-tubes, with 2.0mm and 0.8mm in radius, connect A with B (and C with D).
- Two *t*-tubes, with 0.8mm and 0.4mm in radius, connect A with D (and B with C).
- Filled dots (open dots) are measured at cavity B (D).

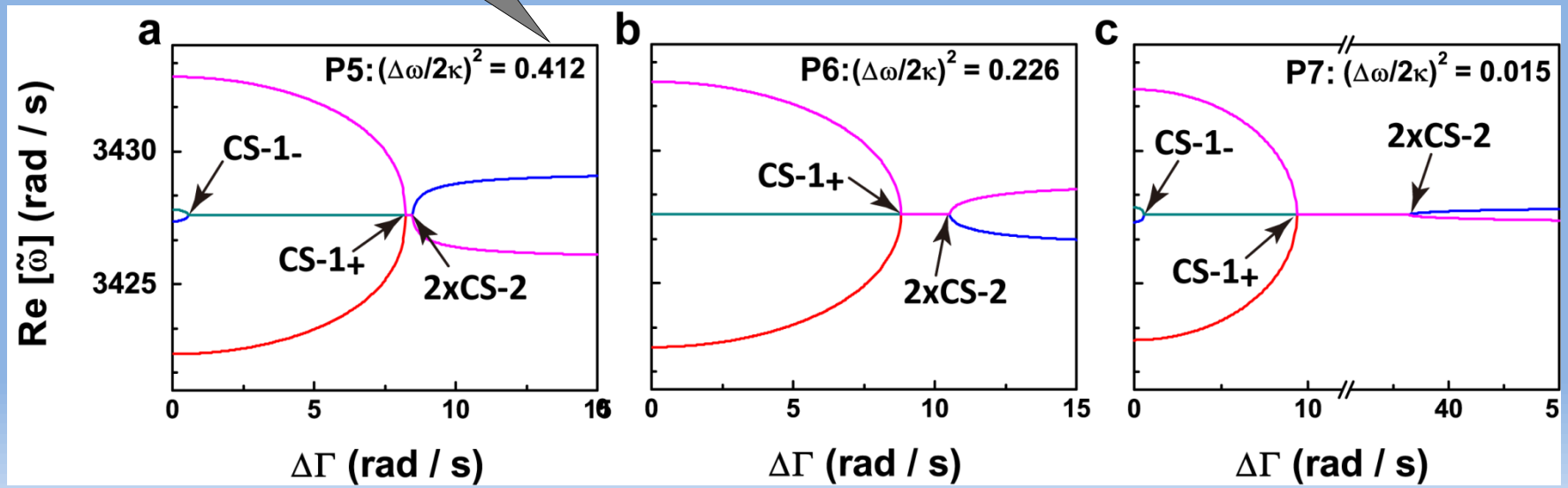
4. Two-state inversion line



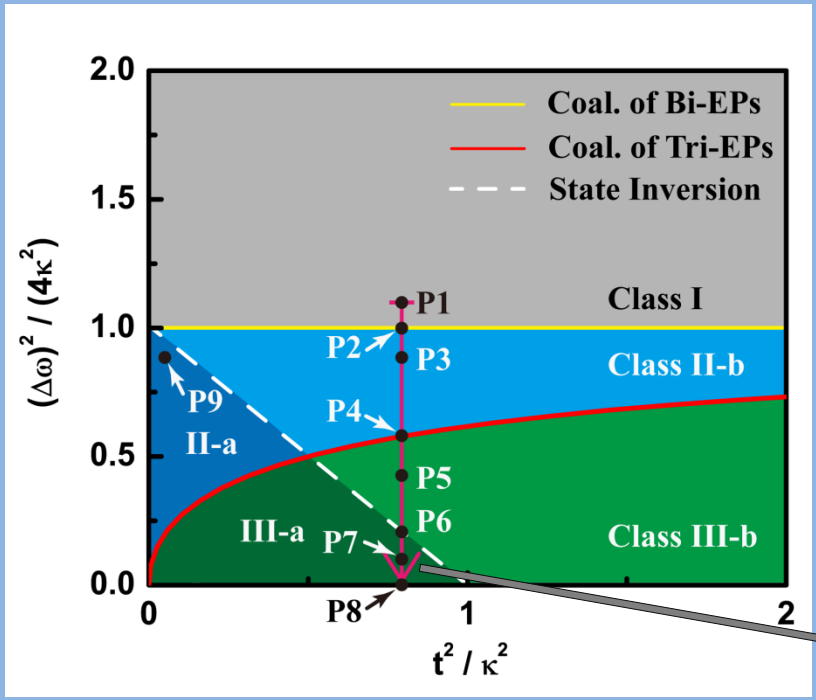
- White dashed line corresponds to degeneracy condition of the two middle states when $\Delta\Gamma = 0$, namely

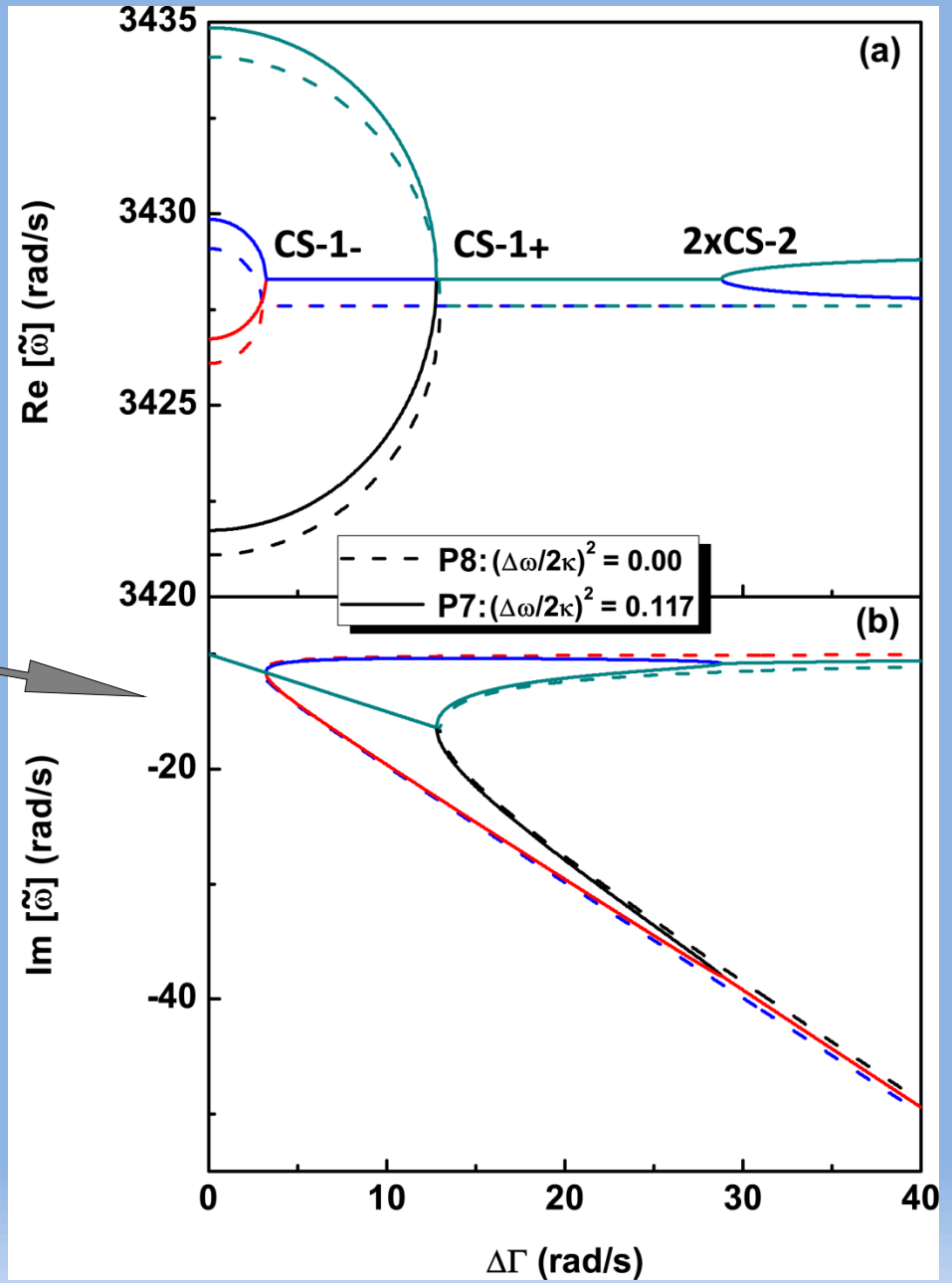
$$4\kappa^2 = 4t^2 + (\Delta\omega)^2$$

- When the system passes this line, the CS-1_ EP disappears firstly and appears again (P5, P6 to P7).
- The chirality of these two CS-1_ EPs is different, as will discuss later.



Class III-a

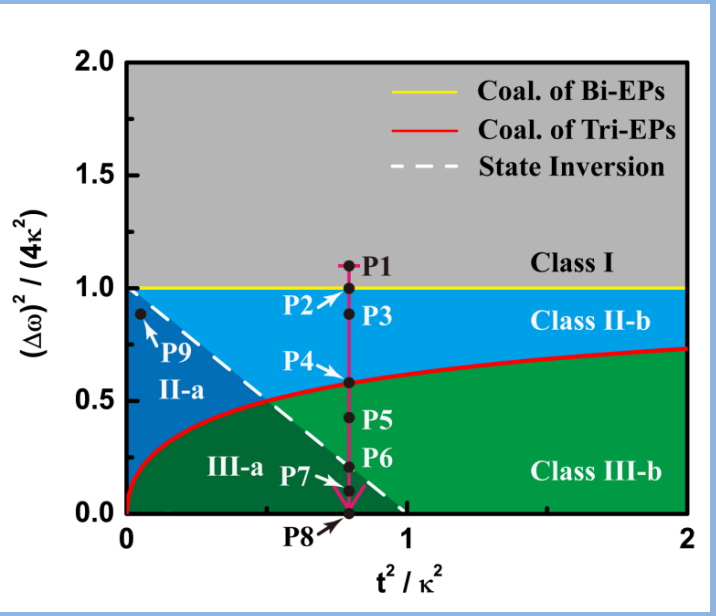




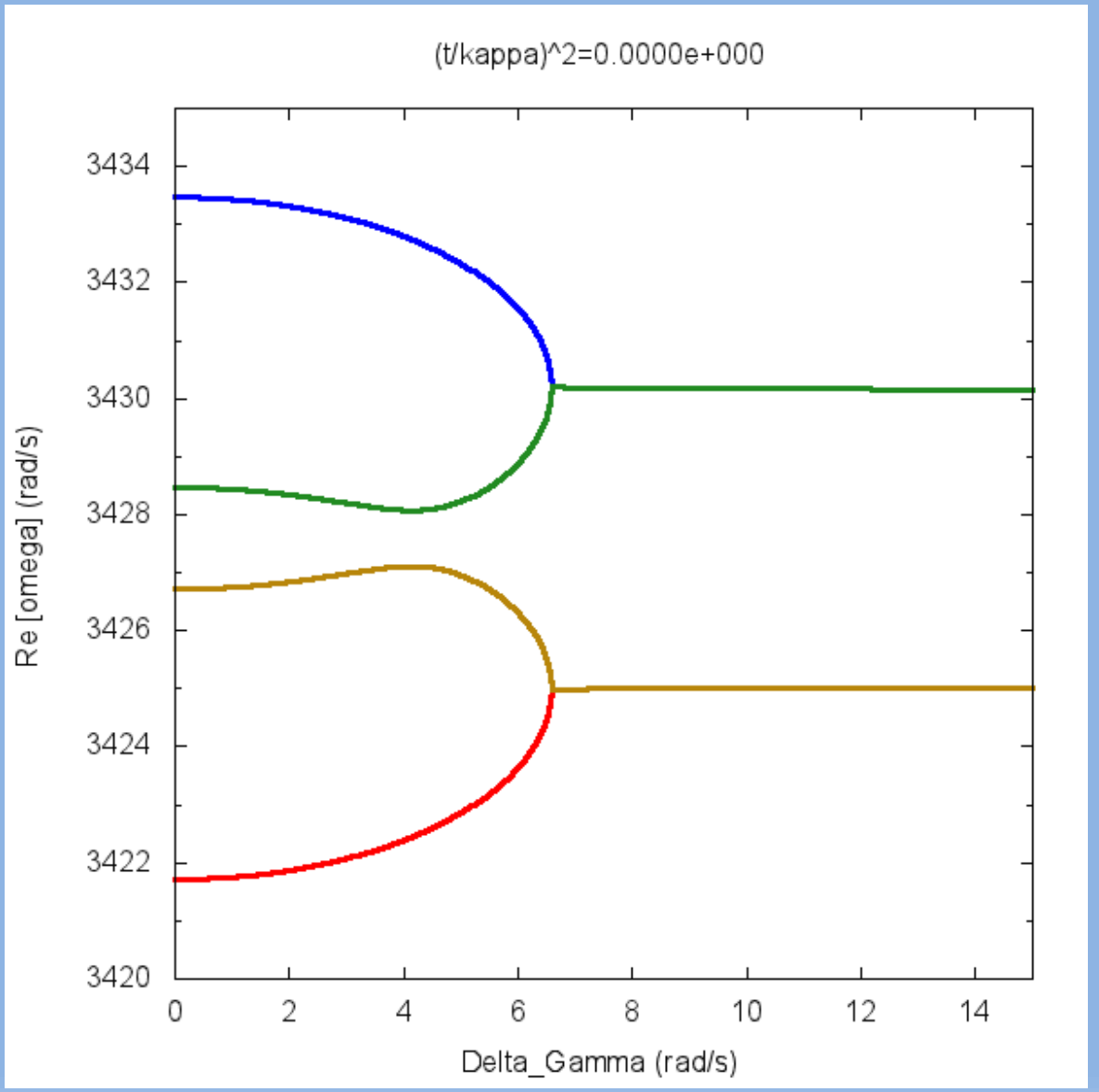
- When $\Delta\omega = 0$, two EPs (CS-$1_{\pm}$) will appear at particular $\Delta\Gamma$ (P8);
- When $\Delta\omega \neq 0$, i.e. Class III-a (P7), additional to two existing CS-1 EPs, another two CS-2 EPs appear at

$$\Delta\Gamma_{CS-2}^2 = 4\kappa^2\left(1+\frac{4t^2}{(\Delta\omega)^2}\right)$$

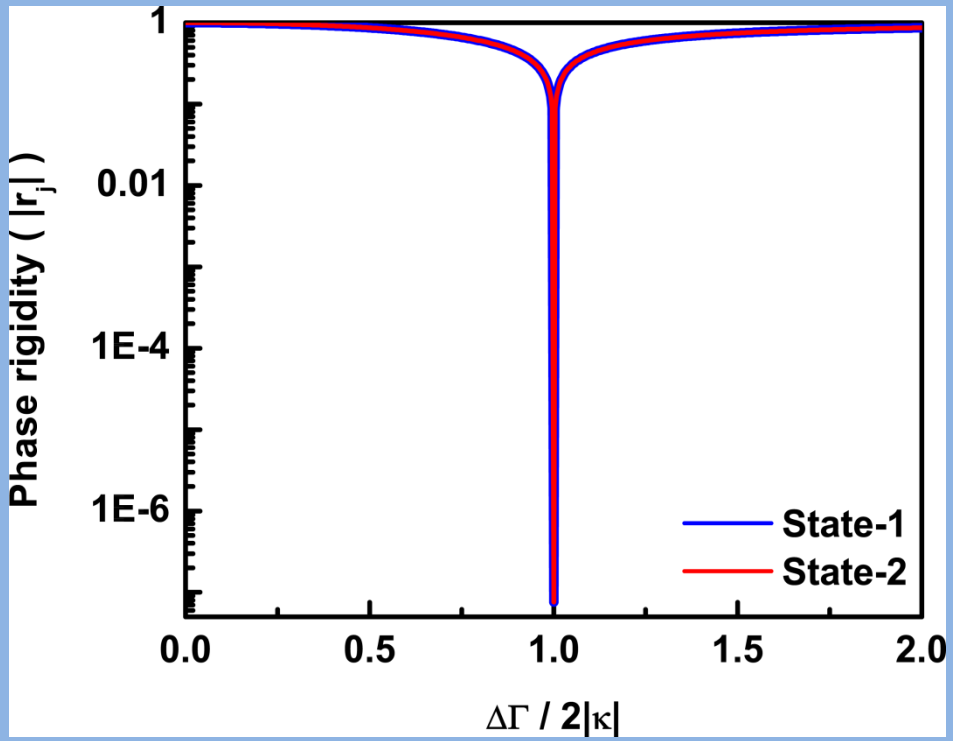
Evolution in the line $(t/\kappa)^2 = 0.7744$



P1. Class I;
P2. Coalescence of Bi-EPs;
P3. Class II-b;
P4. Coalescence of Tri-EPs;
P5. Class III-b;
P6. State inversion;
P7. Class III-a;
P8. Two independent EPs.



A IV. Topological characteristics of singularities



Phase rigidity is defined as: $r_j = \langle \tilde{\phi}_j^R | \tilde{\phi}_j^R \rangle^{-1}$

- Phase rigidity reflects the alignment of states of system with the common environment;
- $r_j = 1$ indicates no alignment, namely Hermitian case;
- $r_j = 0$ indicates fully alignment with scattering state, at the cost of at least one state decouples from environment, namely one defective in eigen-states.

Since only one eigenvector is left at EP, so the **chirality** of EP could be defined as

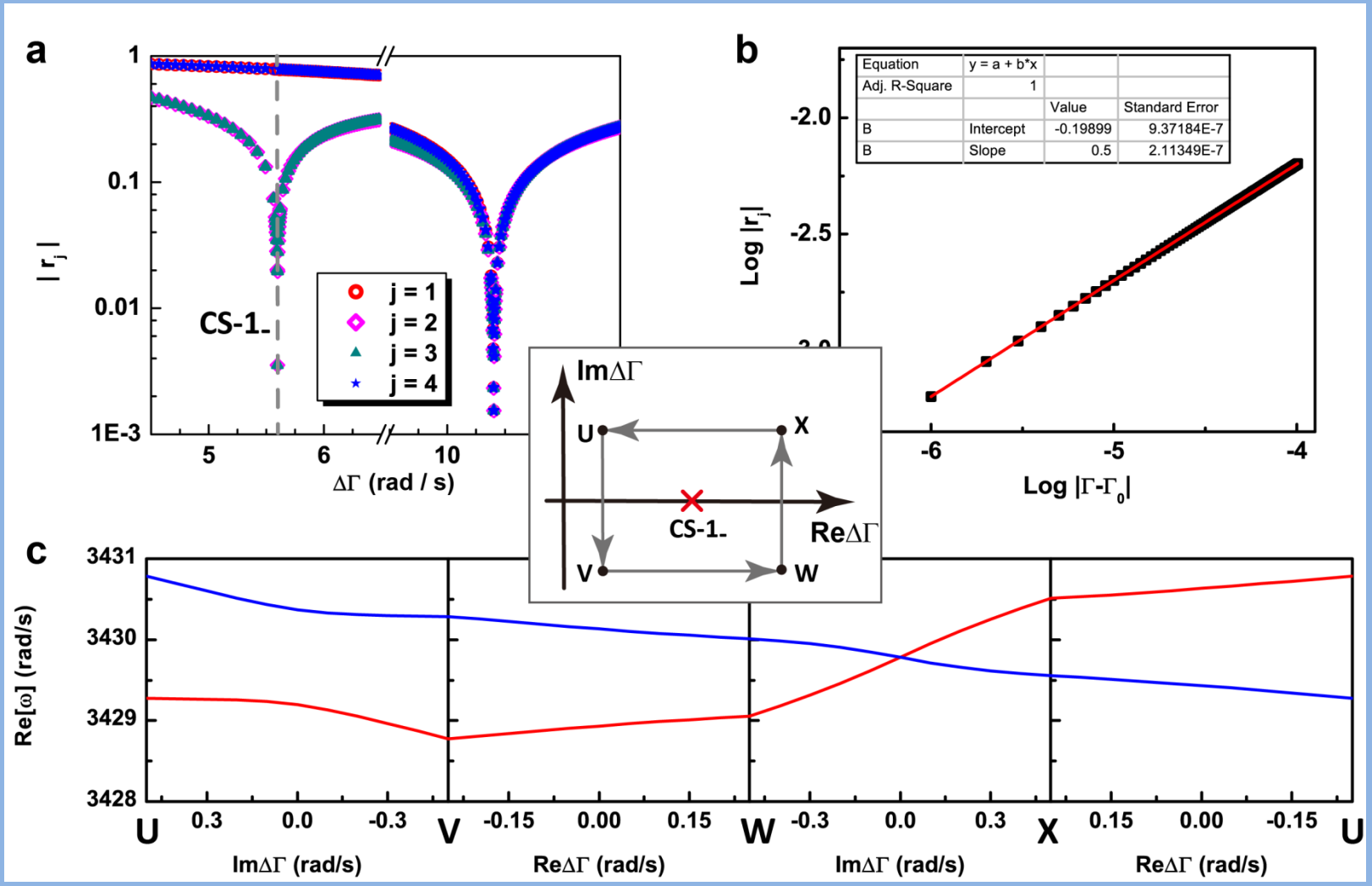
Right chirality: $\phi^R = (-i, 1)/\sqrt{2}$

Left chirality: $\phi^R = (i, 1)/\sqrt{2}$

This property will be reflected by calculating geometric phase for looping around the EP two cycles in the complex $\Delta\Gamma$ plane. Right/left EP will give out geometric phase $\pm\pi$, respectively.

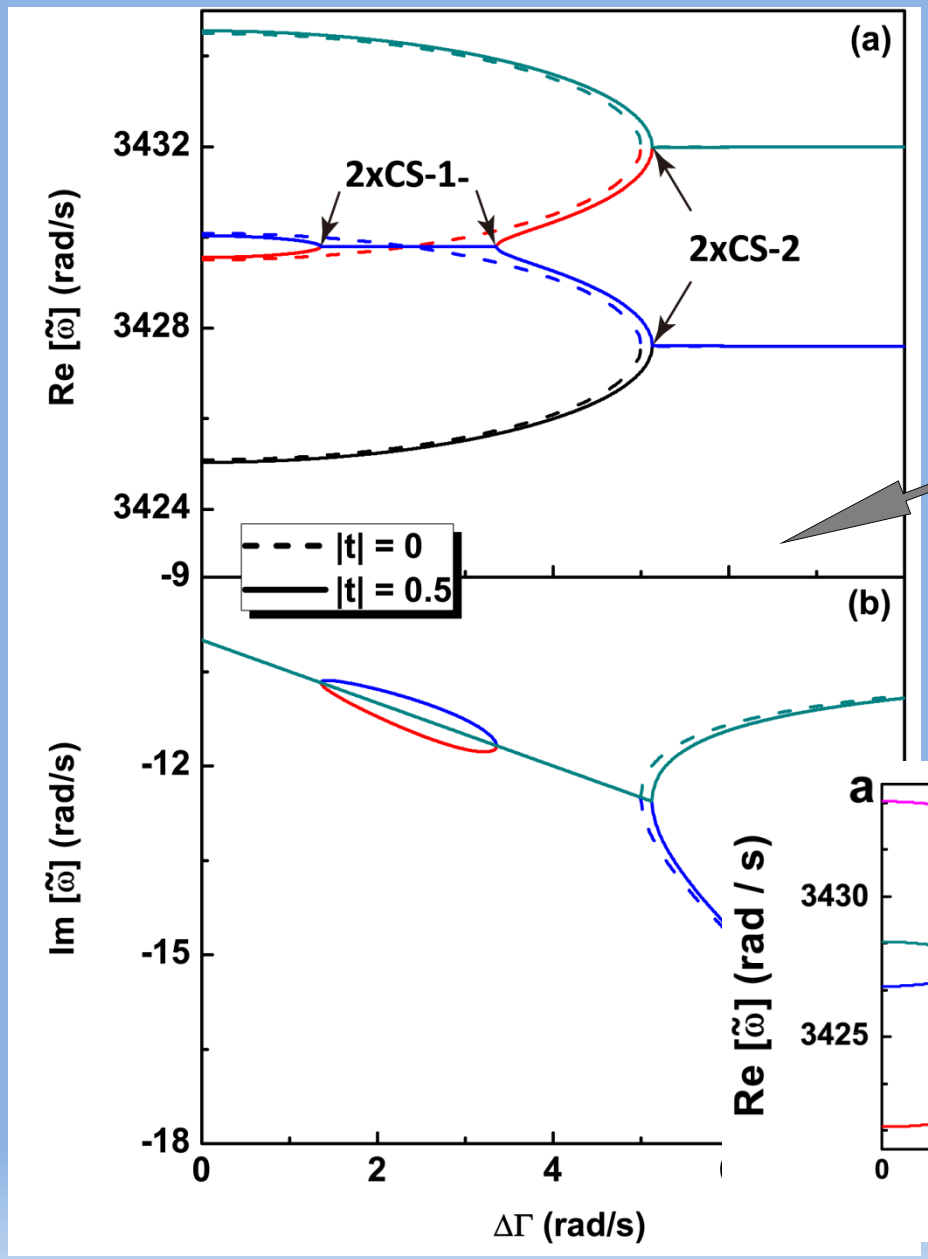
I. Rotter, J. Phys. A: Math. Theor. **42**, 153001 (2009).

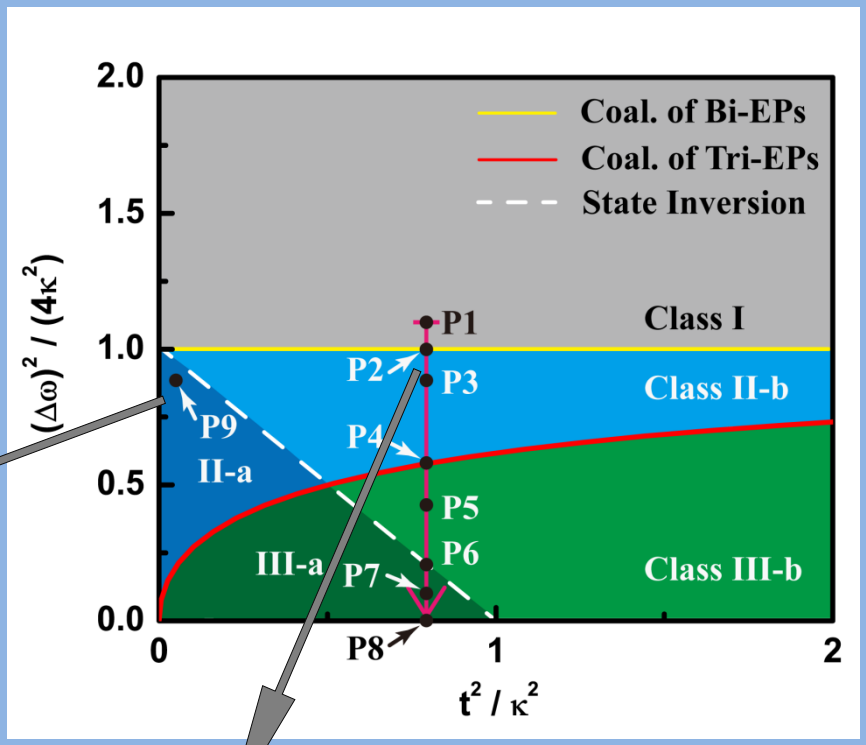
Single exceptional point

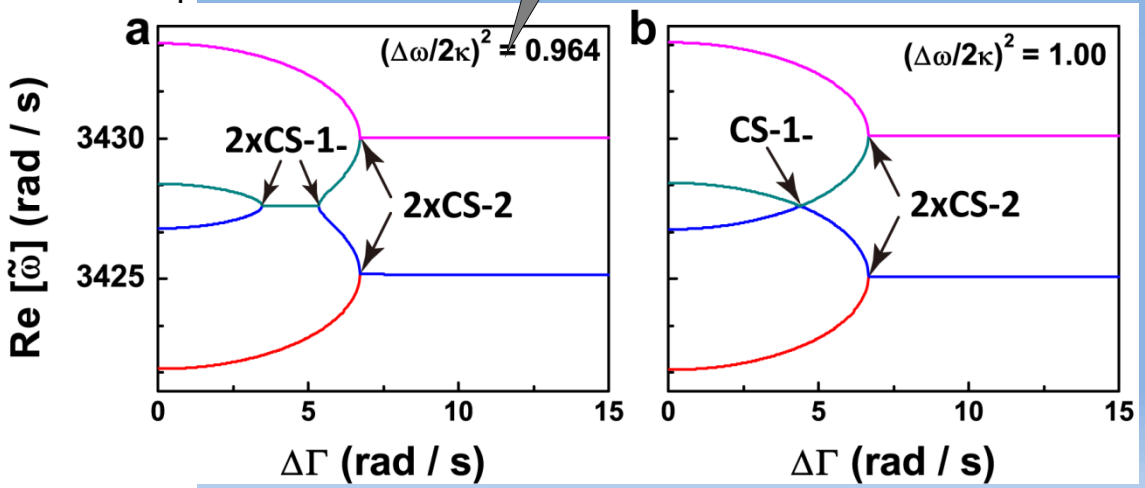


- Log-log fitting of phase rigidity versus $|\Delta\Gamma - \Delta\Gamma_{CS}|$ gives power law index as **1/2** ;
- This index **1/2** could be treated as fractional winding number, i.e. in the complex parameter plane, looping around it **2** circles gives out geometric phase **1** $\times\pi$;
- **Right** and **left** EP gives out geometric phase as $+\pi$ and $-\pi$.

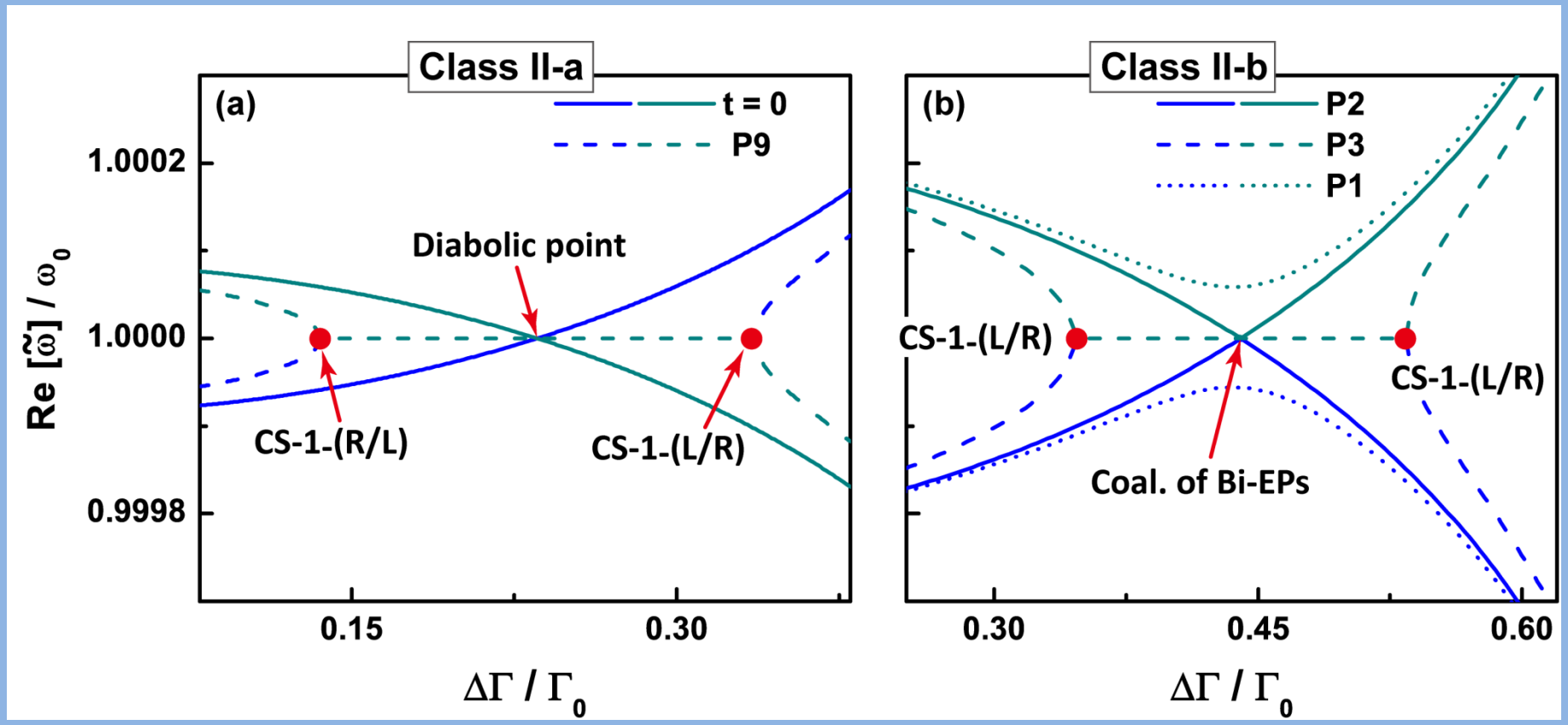
Topological difference of "a" and "b"







Difference between "a" and "b"



$$\frac{1}{2}-\frac{1}{2}=0$$

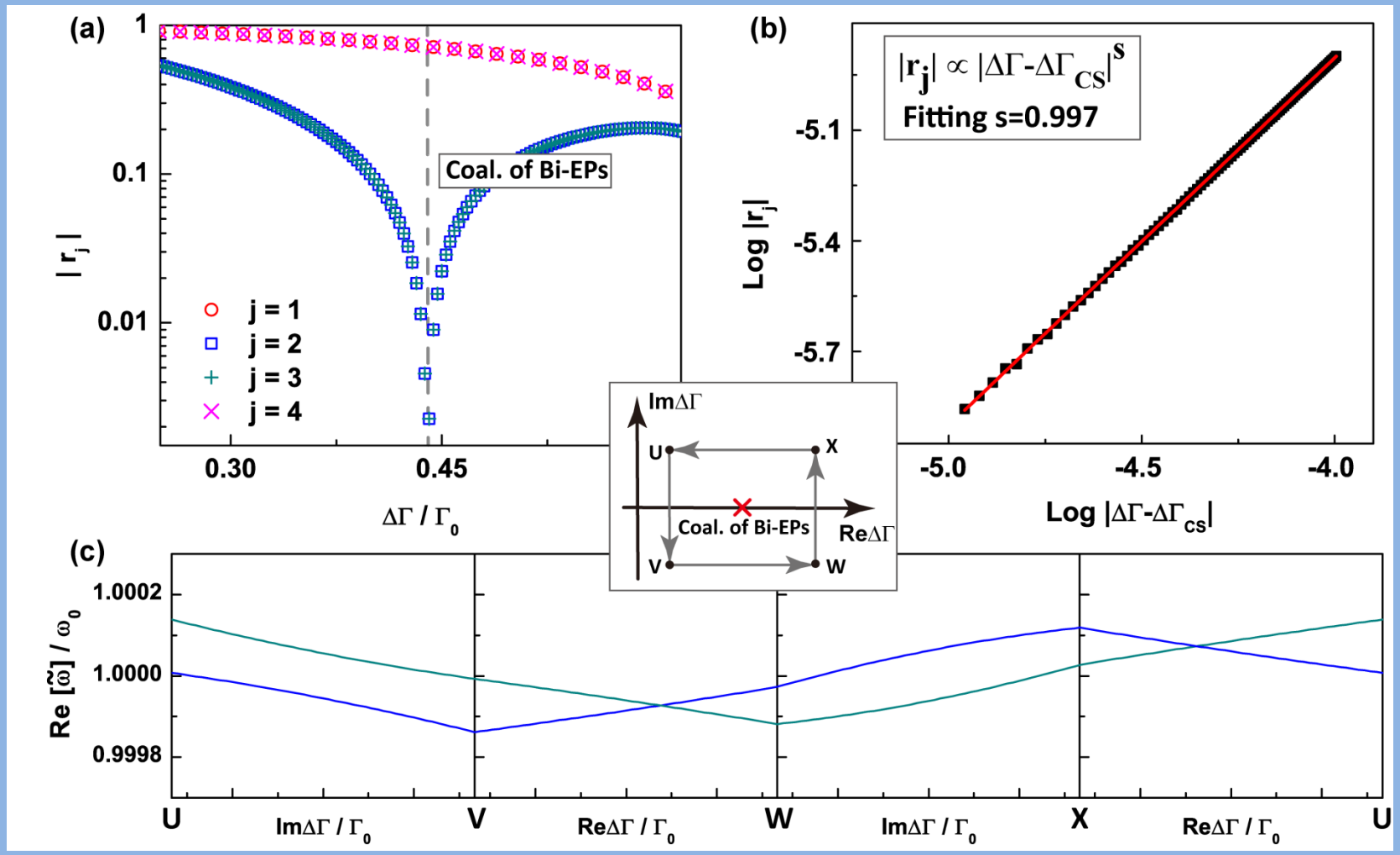
$$\frac{1}{2}+\frac{1}{2}=1$$

- Diabolic point.
- **NO** defective states!

Effective H: $H_{2\times 2}=-\delta\Gamma\sigma_x+i2|t|\sigma_z$

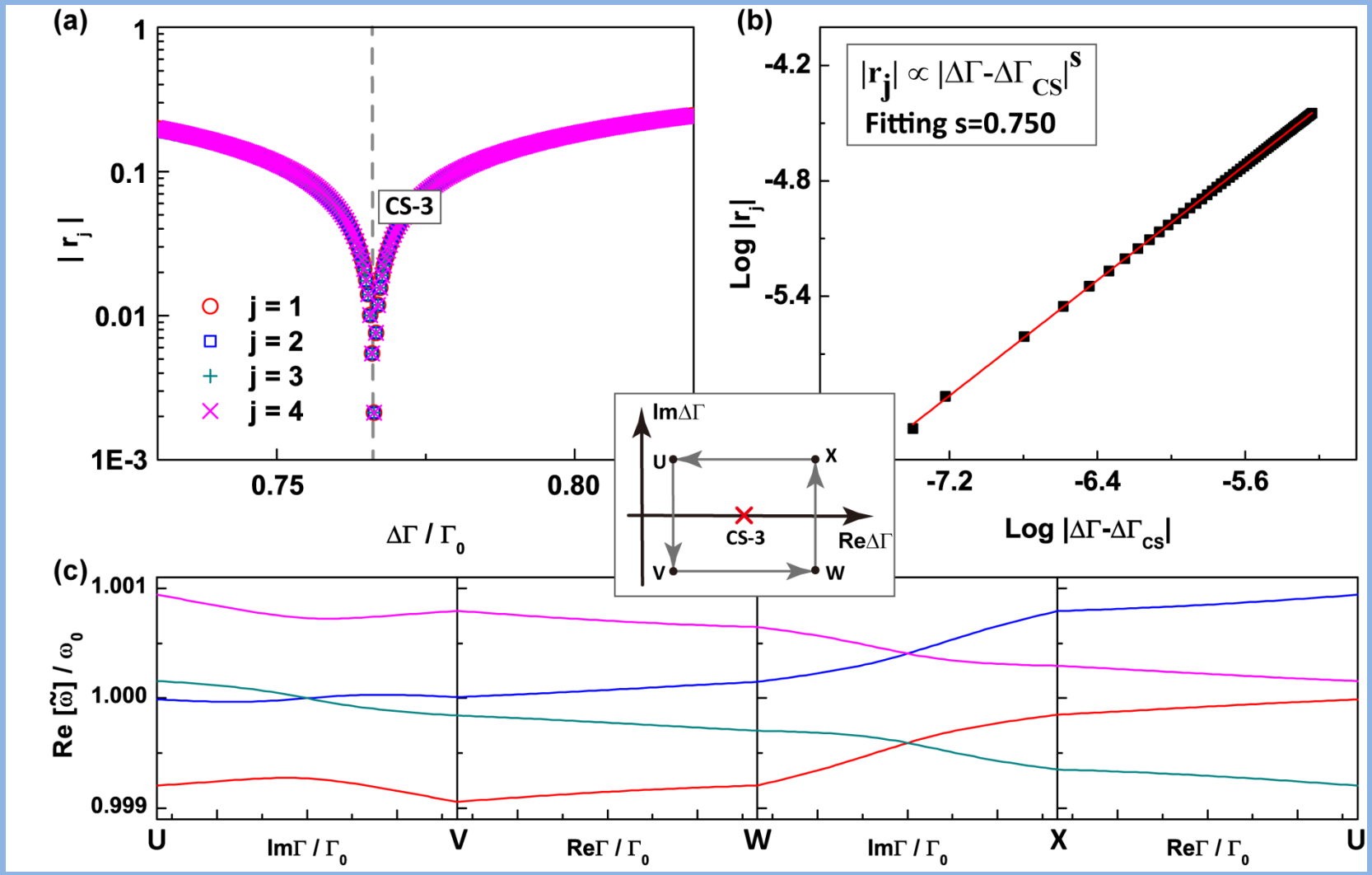
- Higher order singularity!
- One defective state.

Coalescence of Bi-EPs with the same chirality



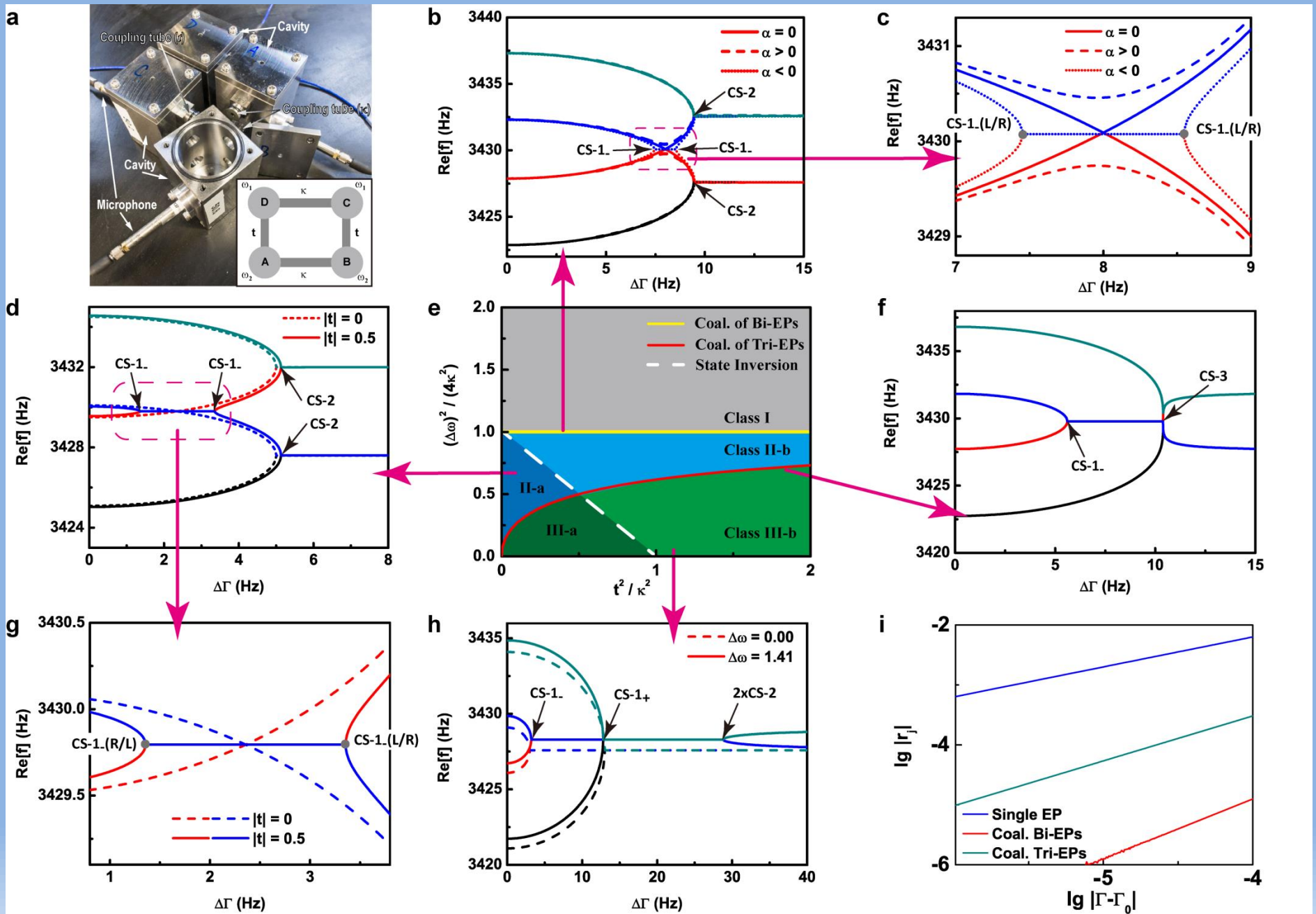
- Log-log fitting of phase rigidity versus $|\Delta\Gamma - \Delta\Gamma_{CS}|$ gives power law index for single EP is **1/1** as shown by **a** and **b**;
- This index **1/1** could be treated as fractional winding number, i.e. in the complex parameter plane, looping around it **1** circles gives out geometric phase **1** $\times\pi$.

Coalescence of Tri-EPs



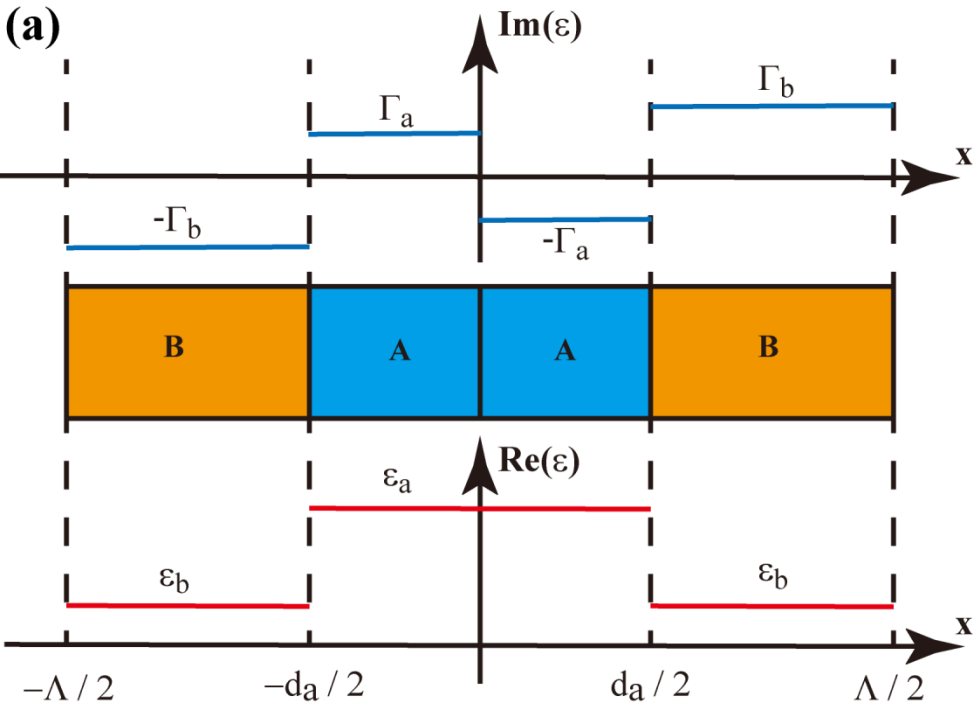
- Log-log fitting of phase rigidity versus $|\Delta\Gamma - \Delta\Gamma_{CS}|$ gives power law index for single EP is **3/4** as shown by **a** and **b**;
- This index **3/4** could be treated as fractional winding number, i.e. in the complex parameter plane, looping around it **4** circles gives out geometric phase **3** $\times\pi$.

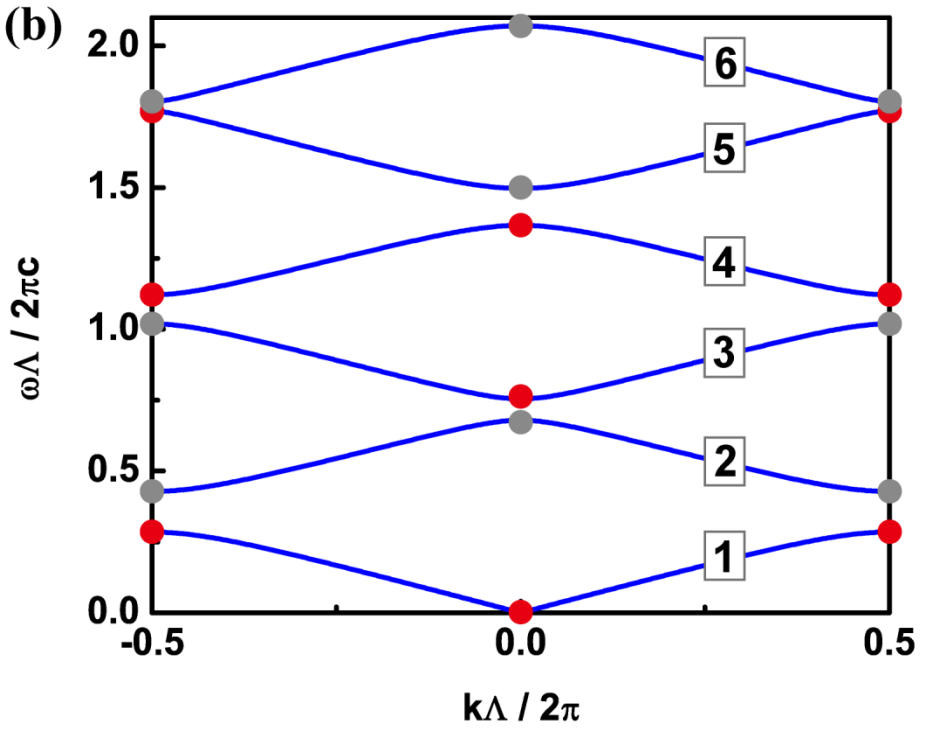
Summary of acoustics part



PC I. Formation of model Hamiltonian

Consider Parity-Time (*PT*) symmetric photonic crystal (left panel) with $\Gamma = \mathrm{Im}[\varepsilon]$, and $\varepsilon(x) = \varepsilon^*(-x)$,





- Parameters: $d_a = 0.42\Lambda, d_b = 0.58\Lambda, \varepsilon_a = 3.8, \varepsilon_b = 1, \mu_a = \mu_b = 1$
- Right panel is band structure for PC without loss or gain. Red (gray) circles denote even (odd) states.

Theoretical formulations

Electric field in the system satisfies the following Helmholtz equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\left(\frac{\omega}{c}\right)^2\varepsilon(x)\right]E_y(x)=0$$

For *PT*-symmetric PC, we could expand the Bloch states $E_{y,nk}^{PT}(x)=\tilde{u}_{nk}(x)e^{ikx}$ by using the set of Bloch states $\left\{E_{y,nk}^{(0)}(x)=u_{nk}(x)e^{ikx}\right\}$ and real eigen-frequencies $\omega_n(k)$ for PC without loss or gain, as

$$\tilde{u}_{nk}(x)=\sum_{n=1}^{+\infty}\alpha_{n,mk}u_{mk}(x)$$

in which $\alpha_{n,mk}$ are expansion coefficients.

Substitute this expansion into wave equation, and we could get Hamiltonian:

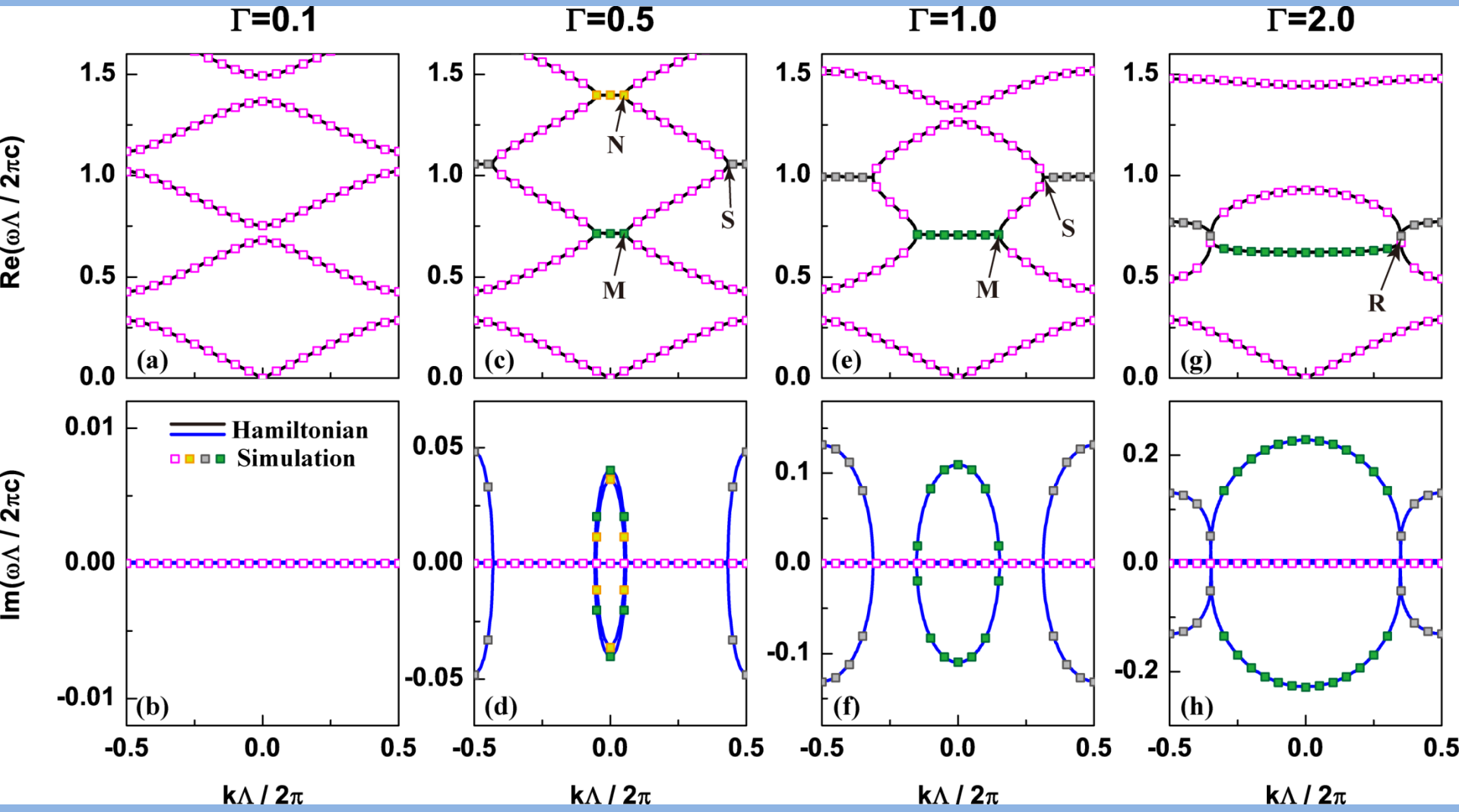
$$H\cdot\tilde{p}=H_1^{-1}\cdot H_2\cdot\tilde{p}=\left(\frac{\omega}{c}\right)^2\tilde{p},$$

in which $\tilde{p}=\left(\alpha_{1k},\cdots,\alpha_{nk},\cdots\right)^T$, $\left(\bar{\varepsilon}_{r,i}\right)_{n'n}=\langle u_{n'k}\,|\,\varepsilon_{r,i}\,|\,u_{nk}\rangle$, *r/i* stands for real/imaginary parts of relative permittivity, $(H_1)_{n'n}=(\bar{\varepsilon}_r)_{n'n}+i(\bar{\varepsilon}_i)_{n'n}$, and $(H_2)_{n'n}=\left(\frac{\omega_n(k)}{c}\right)^2(\bar{\varepsilon}_r)_{n'n}$.

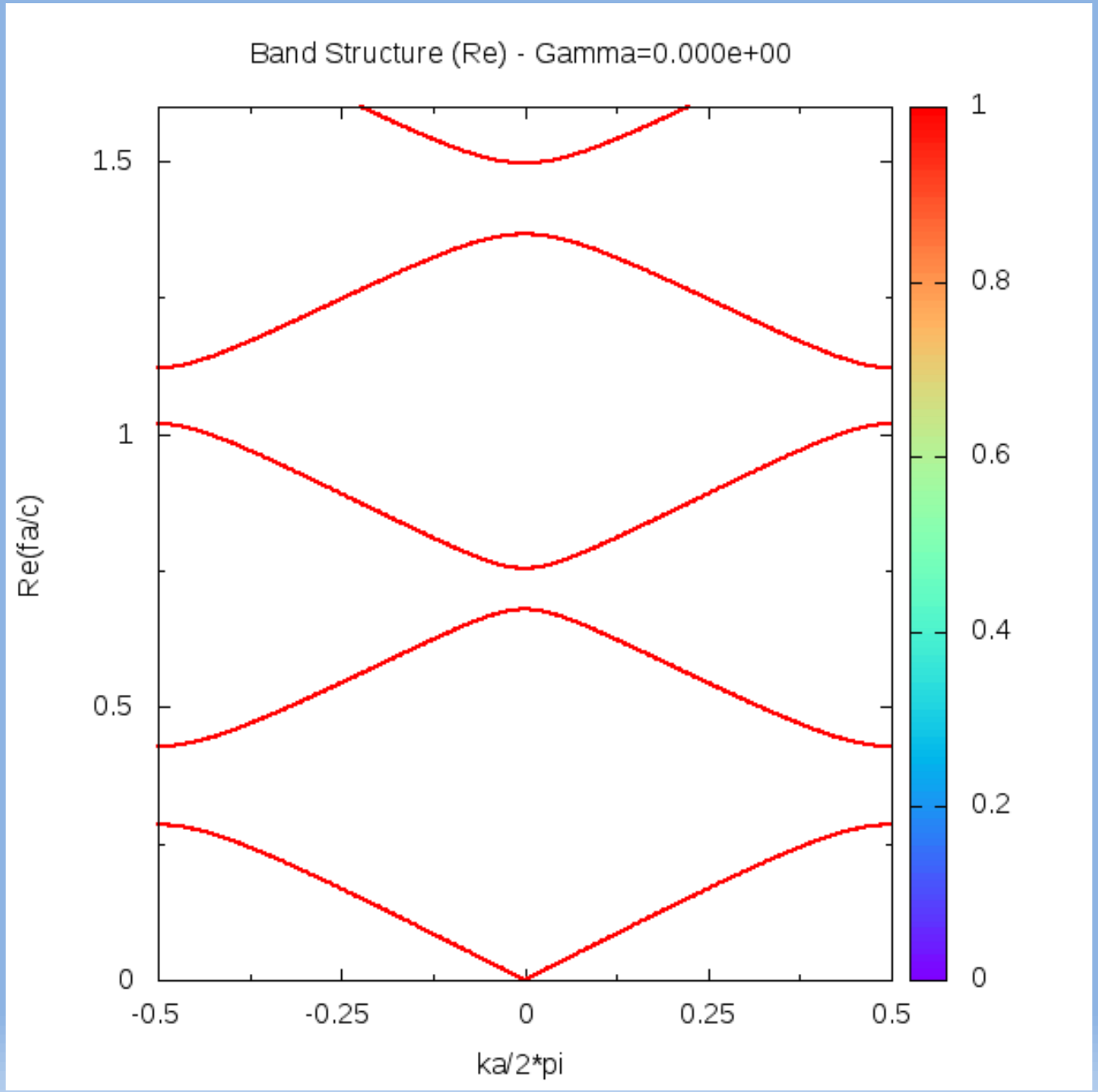
It is worthy to notice that Hamiltonian is **non-Hermitian** due to $(\bar{\varepsilon}_i)_{mn}=(\bar{\varepsilon}_i)_{nm}^*$.

PC II. Complex band and phase diagram

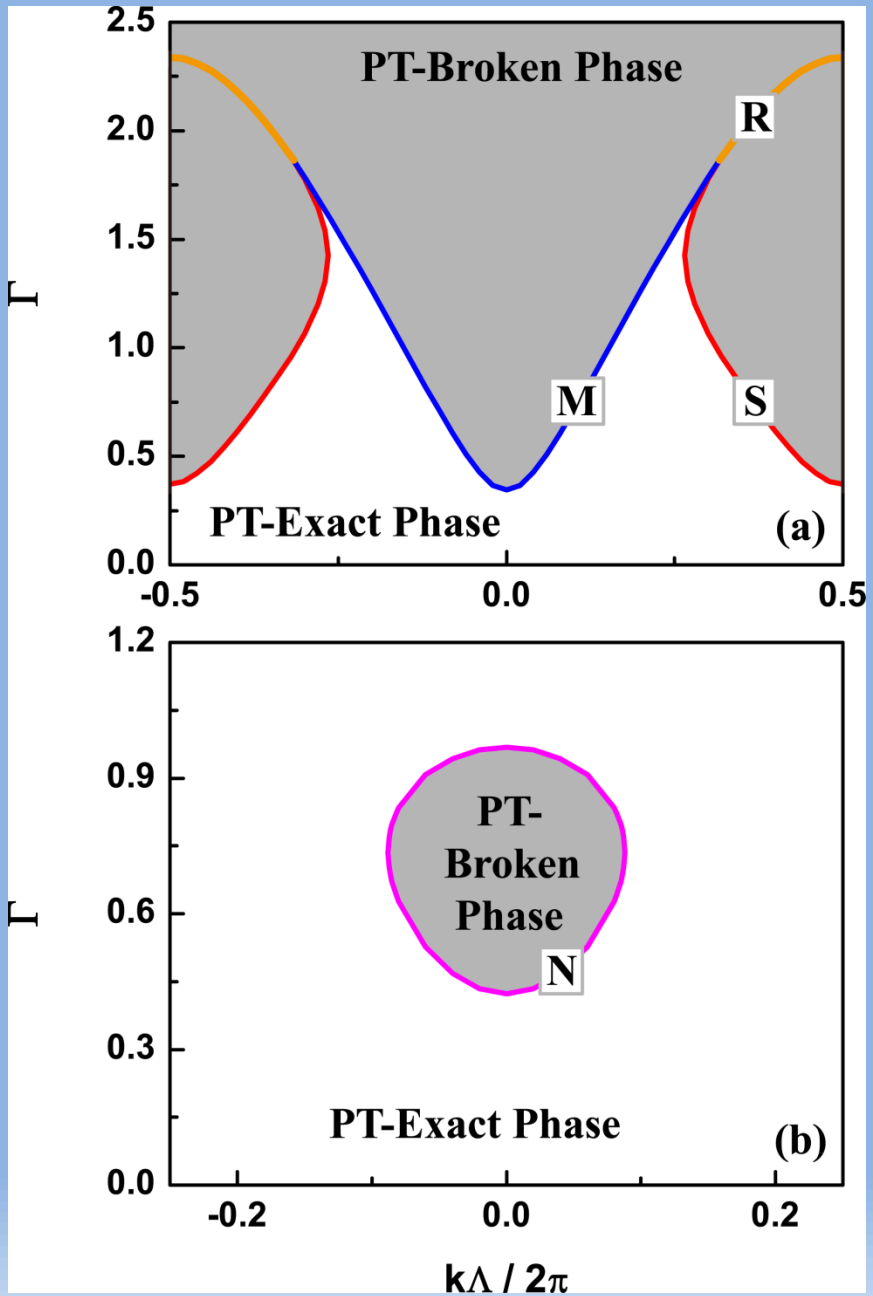
Band structures with complex-frequency



Continuously increase non-Hermiticiy $\Gamma(=\Gamma_a=\Gamma_b)$, and get complex-frequency band structures by using Hamiltonian (solid lines) and Simulation (open dots).


Band Structure (Re) - Gamma=0.000e+00
Re(fa/c)
1.5
1
0.5
0
-0.5
-0.25
0
0.25
0.5
ka/2*pi
1
0.8
0.6
0.4
0.2
0

Two types of PT phase diagram



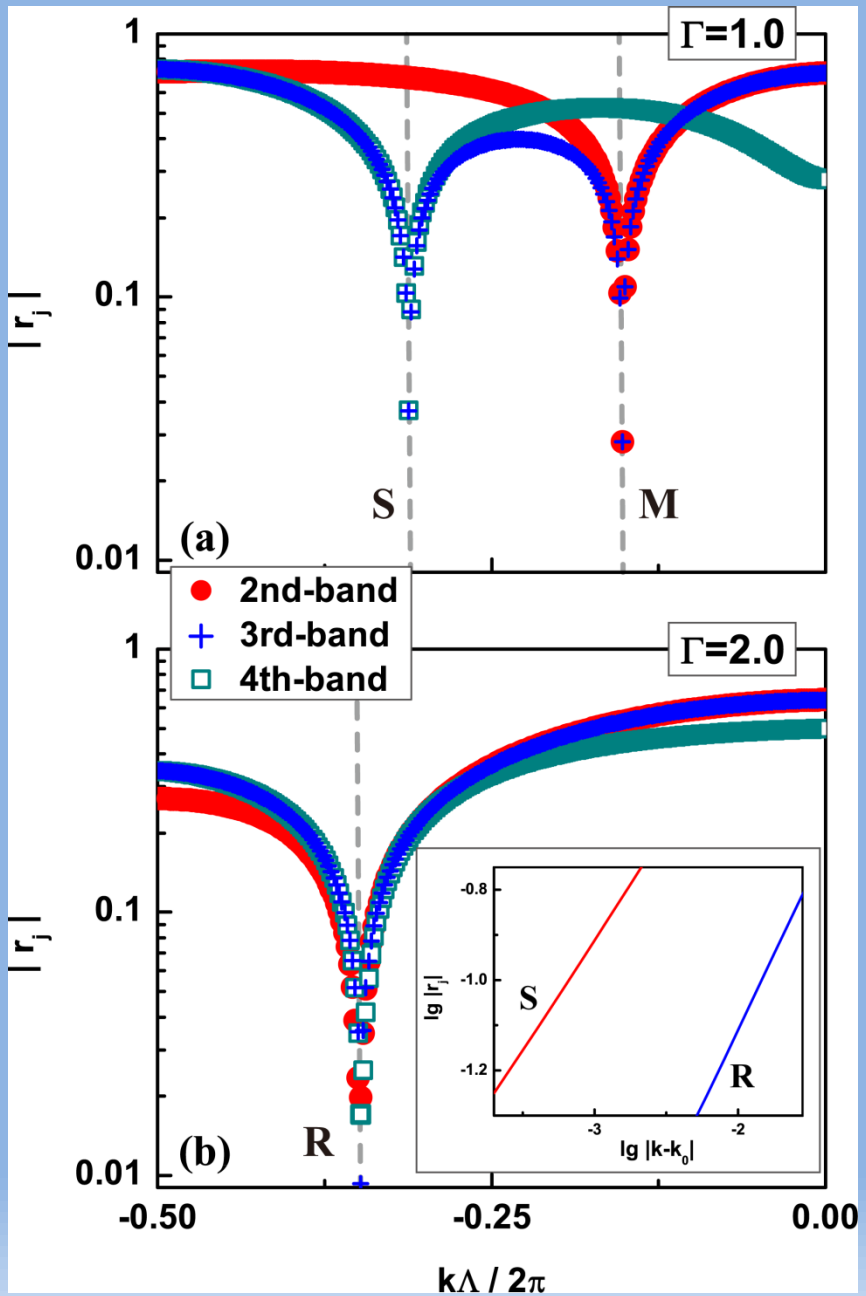
Type I:

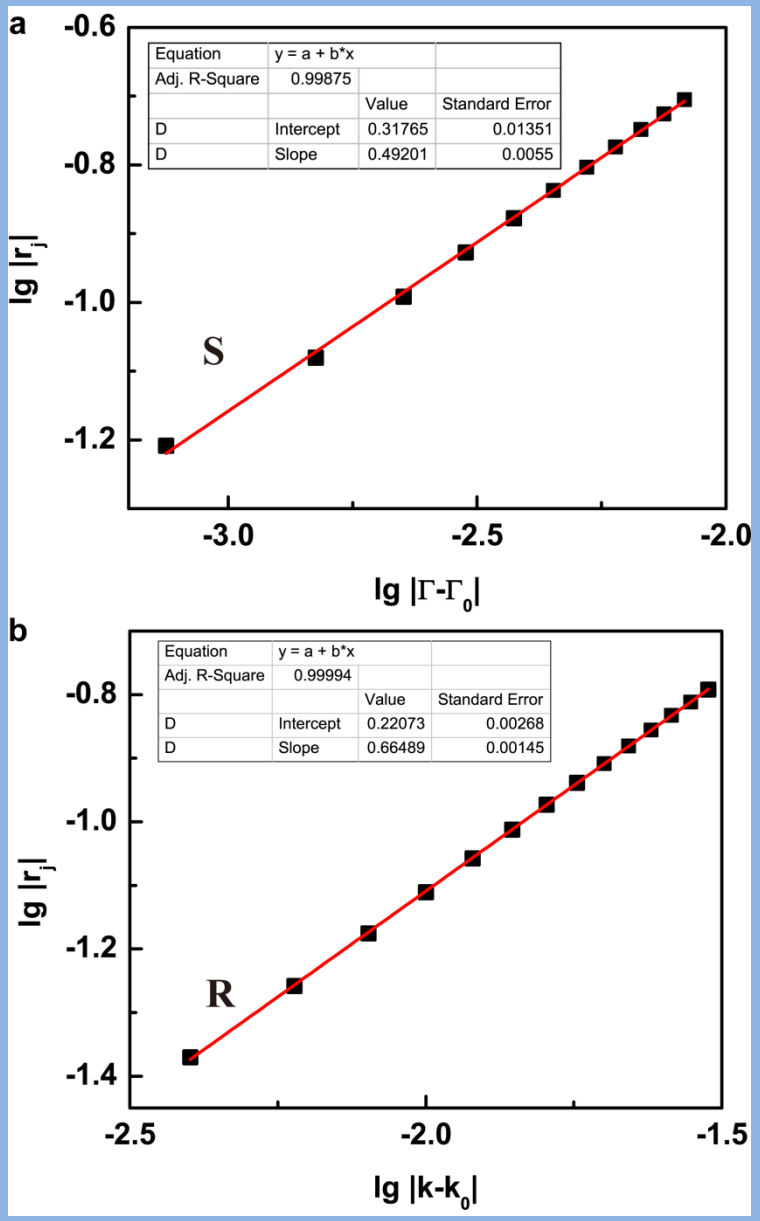
- *M*-EP and *S*-EP appear from different *k* point in the Brillouin zone, but they will kiss each other, namely coalescence of EPs, at some special *k* point in the Brillouin zone;
- The kissing point *R* will move to the Brillouin boundaries after coalescence of *M* and *S* EPs;

Type II:

- It firstly appears at the Brillouin center, and then move towards the Brillouin boundaries. However, the trajectories turn around and move back to the center.
- The loop represents a ring of EPs in the parameter plane.

Phase Rigidity





Power law order of these singularities:

- Single EP (*S* and *M*): **1/2**;
- Coalescence of Bi-EPs (*R*): **2/3**.

Conclusions

- **Multiple exceptional points can coalesce**
  - creating a singularity of higher order with index (N-1)/N;
  - The occurrence of distinctive exceptional point formation patterns in the parameter space have demonstrated in acoustics;
  - Two types of *PT* phase diagrams are found in photonic crystals;
  - Topological properties have been illustrated.

- **Experimental verifications are performed**
  - by using connected cylindrical metallic acoustic cavities.

Thank you!

A: Kun Ding *et al.*, arXiv:1509.06886 (2015).
PC: Kun Ding *et al.*, Phys. Rev. B **92**, 235310 (2015).