# Multiscale Analysis of Wave Propagation and Imaging in Random Media

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- In random media, correlation-based imaging aims at exploiting the information carried by incoherent wave fluctuations when the coherent wave is vanishing.
- Fourth-order moment analysis makes it possible to understand when correlation-based imaging is effective.

# Wave propagation in random media

- Wave equation:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}}) - \Delta_{\vec{\boldsymbol{x}}} u(t,\vec{\boldsymbol{x}}) = F(t,\vec{\boldsymbol{x}}), \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x},z) \in \mathbb{R}^2 \times \mathbb{R}$$

- Time-harmonic source in the plane $z=0$: $F(t,\vec{\boldsymbol{x}}) = \delta(z) f(\boldsymbol{x}) e^{-i\omega t}$.

$\hookrightarrow$ $u(t,\vec{\boldsymbol{x}}) = \hat{u}(\vec{\boldsymbol{x}}) e^{-i\omega t}$ and $\hat{u}$ satisfies

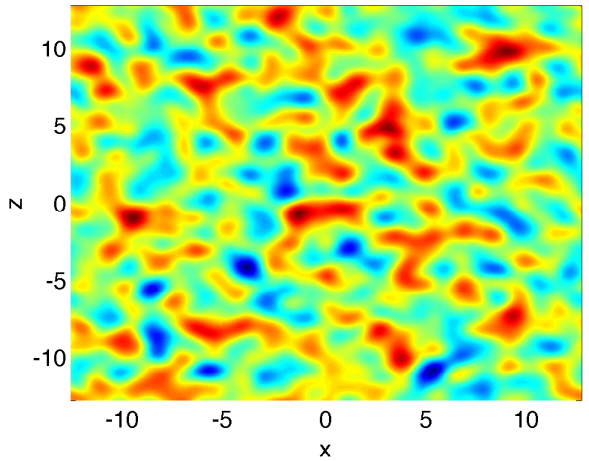
$$\frac{\omega^2}{c^2(\vec{\boldsymbol{x}})}\hat{u}(\vec{\boldsymbol{x}}) + \Delta_{\vec{\boldsymbol{x}}}\hat{u}(\vec{\boldsymbol{x}}) = -\delta(z) f(\boldsymbol{x}), \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x},z) \in \mathbb{R}^2 \times \mathbb{R}$$

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_o^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)$$

$c_o$ is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.




- The statistical properties of the wave field $\hat{u}$ can be studied by multiscale analysis [1,2].

[1] G. Papanicolaou, *SIAM J. Appl. Math.* **21** (1971) 13. [2] J.-P. Fouque et al., Springer, 2007.

## Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation (with $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$):

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_o^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = -\delta(z) f(\boldsymbol{x}).$$

- The function $\hat{\phi}$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}(\boldsymbol{x}, z) = \frac{ic_o}{2\omega} e^{i\frac{\omega z}{c_o}} \hat{\phi}\big(\boldsymbol{x}, z\big)$$

satisfies

$$\partial_z^2 \hat{\phi} + \left(2i\frac{\omega}{c_o}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_o^2}\mu(\boldsymbol{x}, z)\hat{\phi}\right) = 2i\frac{\omega}{c_o}\delta(z) f(\boldsymbol{x}).$$

- If $\mu$ is *mixing* and *smooth*, if

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big), \qquad f(\boldsymbol{x}) \to f\big(\frac{\boldsymbol{x}}{\varepsilon^2}\big),$$

then $\hat{\phi}^\varepsilon$ converges in distribution in $\mathcal{C}^0(\mathbb{R}^+, L^2(\mathbb{R}^2))$ (or $\mathcal{C}^0(\mathbb{R}^+, H^k(\mathbb{R}^2))$) as $\varepsilon \to 0$ to the solution of:

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}')\,\min(z, z')$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$, and $\hat{\phi}(z = 0, \boldsymbol{x}) = f(\boldsymbol{x})$ [1].

[1] J. Garnier et al., *Ann. Appl. Probab.* **19** (2009) 318.

## First-order moments in the paraxial regime

Consider

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z = 0) = f(\boldsymbol{x})$.

- By Itô's formula,

$$\frac{d}{dz}\mathbb{E}[\hat{\phi}] = \frac{ic_o}{2\omega}\Delta_\perp \mathbb{E}[\hat{\phi}] - \frac{\omega^2 \gamma(\mathbf{0})}{8c_o^2}\mathbb{E}[\hat{\phi}]$$

and therefore

$$\mathbb{E}\big[\hat{\phi}(\boldsymbol{x}, z)\big] = \hat{\phi}_{\rm hom}(\boldsymbol{x}, z) \exp\Big( - \frac{\gamma(\mathbf{0})\omega^2 z}{8c_o^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$ and $\hat{\phi}_{\rm hom}$ is the solution in the homogeneous medium.

- Strong damping of the coherent wave.

$\Longrightarrow$ Identification of the *scattering mean free path* $Z_{\rm sca} = \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$ [1].

$\Longrightarrow$ Coherent imaging methods (such as Kirchhoff migration or Reverse-Time migration) fail.

[1] A. Ishimaru, Academic Press, 1978.

## Second-order moments in the paraxial regime

• The mean Wigner transform defined by

$$\mathcal{W}(\boldsymbol{r},\boldsymbol{\xi},z)=\int_{\mathbb{R}^2}\exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{q}\big)\mathbb{E}\left[\hat{\phi}\big(\boldsymbol{r}+\frac{\boldsymbol{q}}{2},z\big)\overline{\hat{\phi}}\big(\boldsymbol{r}-\frac{\boldsymbol{q}}{2},z\big)\right]d\boldsymbol{q},$$

is the angularly-resolved mean wave energy density.

By Itô's formula, it solves a *radiative transport equation*

$$\frac{\partial\mathcal{W}}{\partial z}+\frac{c_o}{\omega}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{r}}\mathcal{W}=\frac{\omega^2}{4(2\pi)^2c_o^2}\int_{\mathbb{R}^2}\hat{\gamma}(\boldsymbol{\kappa})\Big[\mathcal{W}(\boldsymbol{\xi}-\boldsymbol{\kappa})-\mathcal{W}(\boldsymbol{\xi})\Big]d\boldsymbol{\kappa},$$

starting from $\mathcal{W}(\boldsymbol{r},\boldsymbol{\xi},z=0)=\mathcal{W}_0(\boldsymbol{r},\boldsymbol{\xi})$, the Wigner transform of $f$.
$\Longrightarrow$ Identification of the *scattering cross section* $\frac{\omega^2}{4c_o^2}\hat{\gamma}(\boldsymbol{\kappa})$ [1].

• The fields at nearby points are correlated and their correlations contain information about the medium.
$\Longrightarrow$ One should use cross correlations for imaging in random media (CINT [2], ...).

[1] A. Ishimaru, Academic Press, 1978. [2] L. Borcea et al., *Inverse Problems* **21** (2005) 1419.

## Remarks on fourth-order moments

• Physical conjecture: the wave field has Gaussian statistics; therefore we know everything when the first two moments are characterized.
→ the conjecture may be wrong.

• The statistical second-order moments and the mean Wigner transform are not observed directly, only empirical quantities are observed.

• Calculations of fourth-order moments are useful to:

• test the Gaussian conjecture.

• quantify the statistical stability of empirical second-order moments, Wigner transforms, and correlation-based imaging methods.

• implement intensity-correlation-based imaging methods (when only intensities can be measured, as in optics).

## Fourthd-order moments in the random paraxial regime

- Consider

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z=0) = f(\boldsymbol{x})$.

- Let us consider the fourth-order moment:

$$\begin{aligned} M_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{q}_1, \boldsymbol{q}_2, z) &= \mathbb{E}\Big[\hat{\phi}\big(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, z\big)\hat{\phi}\big(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 + \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, z\big) \\ &\quad \times \overline{\hat{\phi}}\big(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, z\big)\overline{\hat{\phi}}\big(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 - \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, z\big)\Big] \end{aligned}$$

By Itô's formula,

$$\frac{\partial M_4}{\partial z} = \frac{ic_o}{\omega}\big(\nabla_{\boldsymbol{r}_1}\cdot\nabla_{\boldsymbol{q}_1} + \nabla_{\boldsymbol{r}_2}\cdot\nabla_{\boldsymbol{q}_2}\big)M_4 + \frac{\omega^2}{4c_o^2}U_4(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2)M_4,$$

with the generalized potential

$$\begin{aligned} U_4(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) &= \gamma(\boldsymbol{q}_2 + \boldsymbol{q}_1) + \gamma(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \gamma(\boldsymbol{r}_2 + \boldsymbol{q}_1) + \gamma(\boldsymbol{r}_2 - \boldsymbol{q}_1) \\ &\quad -\gamma(\boldsymbol{q}_2 + \boldsymbol{r}_2) - \gamma(\boldsymbol{q}_2 - \boldsymbol{r}_2) - 2\gamma(\boldsymbol{0}). \end{aligned}$$

$\Longrightarrow$ One can get a general characterization of the fourth-order moment [1].

[1] J. Garnier et al., *ARMA* **220** (2016) 37.

## Scintillation

Assume that $f(\boldsymbol{x}) = \exp\big(-\frac{|\boldsymbol{x}|^2}{2r_o^2}\big)$.

- The scintillation index:

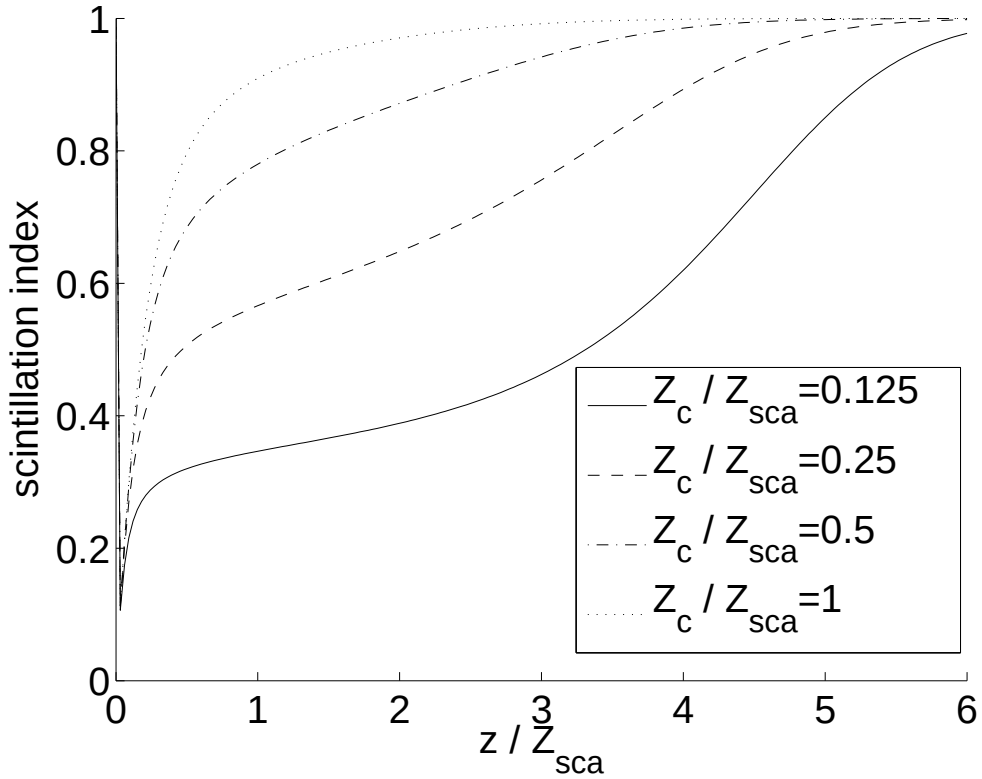
$$S(\boldsymbol{r},z) := \frac{\mathbb{E}\Big[\big|\hat{\phi}(\boldsymbol{r},z)\big|^4\Big] - \mathbb{E}\Big[\big|\hat{\phi}(\boldsymbol{r},z)\big|^2\Big]^2}{\mathbb{E}\Big[\big|\hat{\phi}(\boldsymbol{r},z)\big|^2\Big]^2}$$

satisfies (in the paraxial regime):

$$S(\boldsymbol{r},z) = 1 - \frac{1}{\Big|\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\Big(\frac{\omega^2}{4c_o^2}\int_0^z \gamma\big(\boldsymbol{u}\frac{c_o z'}{\omega r_o}\big)dz' - \frac{|\boldsymbol{u}|^2}{4} + i\boldsymbol{u}\cdot\frac{\boldsymbol{r}}{r_o} + \frac{|\boldsymbol{r}|^2}{r_o^2}\Big)d\boldsymbol{u}\Big|^2}.$$

The physical conjecture is that $S \simeq 1$ when the propagation distance is larger than the scattering mean free path, as it should be for a (complex) Gaussian process.

# Scintillation




Scintillation index at the beam center $S(\mathbf{0}, z)$ as a function of the propagation distance for different values of $Z_{\rm sca} = \frac{8c_o^2}{\omega^2 \gamma(\mathbf{0})}$ and $Z_{\rm c} = \frac{\omega r_o \ell_{\rm c}}{c_o}$. Here $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0}) \exp(-|\boldsymbol{x}|^2/\ell_{\rm c}^2)$.

$\rightarrow$ The physical conjecture that $S \simeq 1$ when $z \gg Z_{\rm sca}$ is true in the random paraxial regime.

## Stability of the Wigner transform of the field

- The Wigner transform

$$W(\boldsymbol{r},\boldsymbol{\xi},z) := \int_{\mathbb{R}^2} \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{q}\big)\hat{\phi}\big(\boldsymbol{r}+\frac{\boldsymbol{q}}{2},z\big)\overline{\hat{\phi}}\big(\boldsymbol{r}-\frac{\boldsymbol{q}}{2},z\big)d\boldsymbol{q}$$

is not statistically stable (i.e. standard deviation > mean).

- Let us consider the smoothed Wigner transform (for $r_{\rm s},\xi_{\rm s}>0$):

$$W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) = \frac{1}{(2\pi)^2 r_{\rm s}^2\xi_{\rm s}^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} W(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{\xi}-\boldsymbol{\xi}',z)\exp\Big(-\frac{|\boldsymbol{r}'|^2}{2r_{\rm s}^2}-\frac{|\boldsymbol{\xi}'|^2}{2\xi_{\rm s}^2}\Big)d\boldsymbol{r}'d\boldsymbol{\xi}'.$$

Its coefficient of variation:

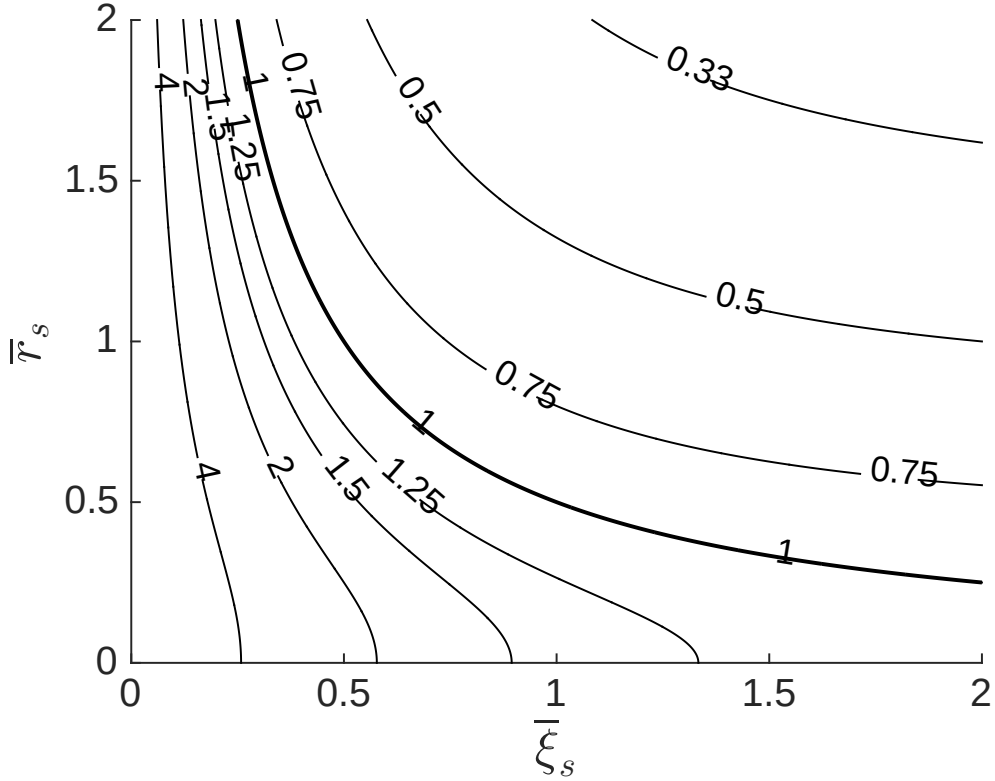
$$C_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) := \frac{\sqrt{\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)^2]-\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)]^2}}{\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)]}$$

determines its statistical stability.

$\hookrightarrow$ Analysis of high-order moments of $\hat{\phi}$ [1].

[1] J. Garnier et al., *ARMA* **220** (2016) 37.

## Stability of the Wigner transform of the field



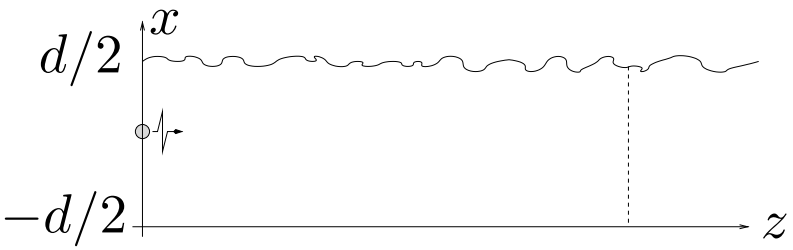

Contour levels of the coefficient of variation of the smoothed Wigner transform. Here $\overline{r}_{\rm s} = r_{\rm s}/\rho$, $\overline{\xi}_{\rm s} = \xi_{\rm s}\rho$, and $\rho = \rho(z; \omega, r_o, \ell_c, Z_{\rm sca})$.

→ This result makes it possible to achieve optimal trade-off between stability and resolution for correlation-based imaging [1,2].

Example: ultrasonic non-destructive testing in concrete.

[1] L. Borcea et al., *Inverse Problems* **27** (2011) 085004. [2] J. Garnier et al., *ARMA* **220** (2016) 37.

# Wave propagation in random waveguides




Wave propagation in two-dimensional waveguides:

$$\Big[(\partial_x^2+\partial_z^2)+k^2\mathrm{n}^2(x,z)\Big]\hat{u}(x,z)=\delta(z)f_{\mathrm{s}}(x)$$

- Ideal waveguide:

$$\mathrm{n}^{(0)}(x)=\begin{cases} n & \text{if } x\in(-d/2,d/2),\\ 1 & \text{otherwise,}\end{cases}$$

where $n>1$ is the relative index of the core and $d>0$ is its diameter.

- Type I perturbation: the index of refraction within the core is randomly perturbed:

$$\mathrm{n}^{(\varepsilon)}(x,z)=\begin{cases} n+\varepsilon\nu(x,z) & \text{if } x\in(-d/2,d/2) \text{ and } z\in(0,L/\varepsilon^2),\\ 1 & \text{otherwise.}\end{cases}$$

- Type II perturbation: the boundaries of the core are randomly perturbed:

$$\mathrm{n}^{(\varepsilon)}(x,z)=\begin{cases} n & \text{if } x\in\big(-d/2+\varepsilon d\nu_-(z),d/2+\varepsilon d\nu_+(z)\big) \text{ and } z\in(0,L/\varepsilon^2),\\ 1 & \text{otherwise.}\end{cases}$$

# Ideal waveguide

• The Helmholtz operator $\partial_x^2 + k^2 \mathrm{n}^{(0)}(x)$ has spectrum $(-\infty, k^2) \cup \{\beta_{N-1}^2, \dots, \beta_0^2\}$.

- the $N$ modal wavenumbers $\beta_j$ are positive and $k^2 < \beta_{N-1}^2 < \cdots < \beta_0^2 < n^2 k^2$.
- $\phi_j$, $j = 0, \dots, N-1$, are the eigenfunctions associated to $\beta_j^2$.
- $\phi_\gamma^{(t)}$, $t \in \{e, o\}$ are the generalized eigenfunctions associated to $\gamma \in (-\infty, k^2)$.

• $\hat{u}$ can be expanded on the complete set of orthonormal eigenmodes:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} u_j(z)\phi_j(x) + \sum_{t\in\{e,o\}} \int_{-\infty}^{k^2} u_\gamma^{(t)}(z)\phi_\gamma^{(t)}(x)d\gamma$$

The mode amplitudes satisfy for $z \neq 0$:

$$\partial_z^2 u_j + \beta_j^2 u_j = 0, \quad j = 0, \dots, N-1, \qquad \partial_z^2 u_\gamma^{(t)} + \gamma u_\gamma^{(t)} = 0, \quad \gamma \in (-\infty, k^2)$$

Therefore, for $z > 0$:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \underbrace{\frac{a_{j,\mathrm{s}}}{\sqrt{\beta_j}} e^{i\beta_j z}\phi_j(x)}_{\text{guided mode}} + \sum_{t\in\{e,o\}} \int_0^{k^2} \underbrace{\frac{a_{\gamma,\mathrm{s}}^{(t)}}{\gamma^{1/4}} e^{i\sqrt{\gamma}z}\phi_\gamma^{(t)}(x)}_{\text{radiating mode}} d\gamma + \int_{-\infty}^{0} \underbrace{\frac{a_{\gamma,\mathrm{s}}^{(t)}}{|\gamma|^{1/4}} e^{-\sqrt{|\gamma|}z}\phi_\gamma^{(t)}(x)}_{\text{evanescent mode}} d\gamma$$

The coefficients $a_{\cdot,\mathrm{s}}$ are constant and determined by the source.

## Random waveguide

- $\hat{u}$ can be expanded on the complete set of eigenmodes of the ideal waveguide:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} u_j(z)\phi_j(x) + \sum_{t\in\{e,o\}} \int_{-\infty}^{k^2} u_\gamma^{(t)}(z)\phi_\gamma^{(t)}(x)d\gamma$$

$\rightarrow$ $u_j$, $u_\gamma^{(t)}$ satisfy coupled equations.

- For large propagation distances:

$$\hat{u}(x,\frac{z}{\varepsilon^2}) \simeq \sum_{j=0}^{N-1} c_p(z)e^{i\beta_j\frac{z}{\varepsilon^2}}\phi_j(x)$$

and the moments of $c_p$ satisfy closed-form equations.
This result is obtained in the limit $\varepsilon \to 0$, in distribution, in $\mathcal{C}^0(\mathbb{R}^+, L^2_{weak}(\mathbb{R}))$.

## First- and second-order moments in random waveguides

- $\mathbb{E}[c_p]$ decays exponentially with $z$ (with a mode-dependent scattering mean free path $\ell_p$),
- $(\mathbb{E}[|c_p|^2])_{p=1}^N$ satisfies the coupled system

$$\frac{d}{dz}\mathbb{E}[|c_p|^2] = \sum_{q\neq p}\Gamma_{pq}(\mathbb{E}[|c_q|^2] - \mathbb{E}[|c_p|^2]) - \Lambda_p\mathbb{E}[|c_p|^2]$$

where

$$\Gamma_{jl} = \frac{k^4}{2\beta_j\beta_l}\int_0^\infty \mathcal{R}_{jl}(z)\cos\big((\beta_l-\beta_j)z\big)dz,$$

$$\mathcal{R}_{jl}(z) = \begin{cases} \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_j\phi_l(x)\mathbb{E}[\nu(x,0)\nu(x',z)]\phi_j\phi_l(x')dxdx' & \text{type I} \\ (n^2-1)^2d^2\left[\phi_j^2\phi_l^2\big(-\tfrac{d}{2}\big)\mathbb{E}[\nu_-(0)\nu_-(z)] + \phi_j^2\phi_l^2\big(\tfrac{d}{2}\big)\mathbb{E}[\nu_+(0)\nu_+(z)]\right] & \text{type II} \end{cases}$$

$$\Lambda_j = \int_0^{k^2}\frac{k^4}{2\sqrt{\gamma}\beta_j}\sum_{t\in\{e,o\}}\int_0^\infty \mathcal{R}^{(t)}_{j,\gamma}(z)\cos\big((\sqrt{\gamma}-\beta_j)z\big)dzd\gamma$$

$\rightarrow$ exchange of energy between the guided modes,

$\rightarrow$ loss of energy towards the radiating modes.

## Fourth-order moments in random waveguides

- The moments

$$R_{jl}(z) = \mathbb{E}\big[|c_j|^2(z)|c_l|^2(z)\big], \quad j,l = 0,\ldots,N-1,$$

satisfy the closed equations

$$\partial_z R_{jj} = -2\Lambda_j R_{jj} + \sum_{n\neq j} \Gamma_{jn}(4R_{jn} - 2R_{jj}),$$

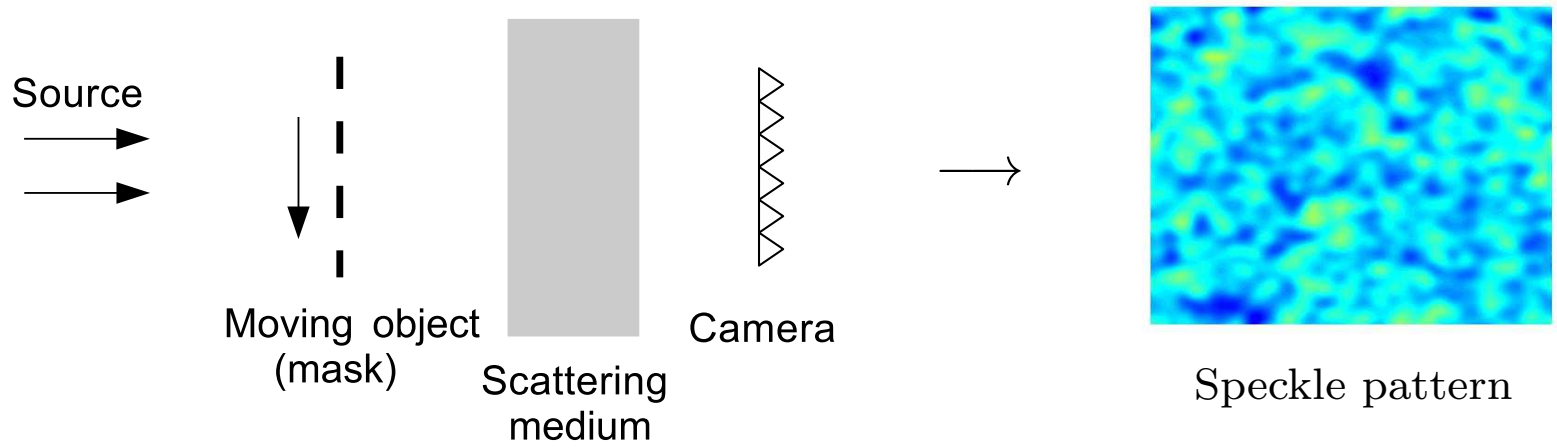
$$\partial_z R_{jl} = -(2\Gamma_{jl} + \Lambda_j + \Lambda_l)R_{jl} + \sum_{n\neq l} \Gamma_{ln}(R_{jn} - R_{jl}) + \sum_{n\neq j} \Gamma_{jn}(R_{nl} - R_{jl}), \quad j \neq l.$$

- Analysis of the linear system: $\mathrm{Var}(|c_p|^2)/\mathbb{E}[|c_p|^2]^2$ grows exponentially with $z$ !
- Remark: When $N \to \infty$, the exponential growth rate goes to zero.
- This may happen in underwater acoustics (Pekeris waveguide).
- Correlation-based imaging is challenging in such conditions.

## Remarks on fourth-order moments

- Calculations of fourth-order moments are useful to:
- test the Gaussian conjecture.
- quantify the statistical stability of empirical second-order moments, Wigner transforms, and correlation-based imaging methods.
- implement intensity-correlation-based imaging methods (when only intensities can be measured, as in optics).

## Speckle intensity correlation imaging through a scattering medium

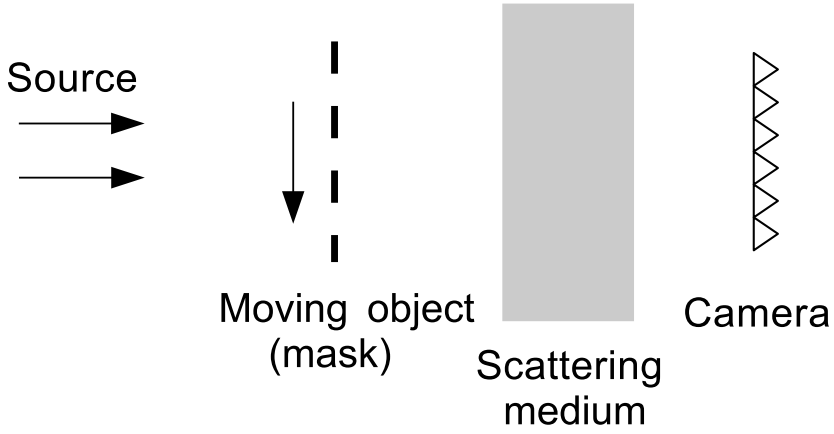



Experimental set-up [1]

- The light source is a time-harmonic plane wave.
- The object to be imaged is a mask that can be shifted transversally.
- For each position of the object the spatial intensity of the transmitted field (speckle pattern) can be recorded by the camera.

[1] J. A. Newmann et al., *Phys. Rev. Lett.* **113** (2014) 263903.

# Speckle intensity correlation imaging through a scattering medium




- The field just after the mask (in the plane $z = 0$) is (for a transverse shift $\boldsymbol{r}$):

$$f_{\boldsymbol{r}}(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{r}),$$

where $f$ is the indicator function of the mask.

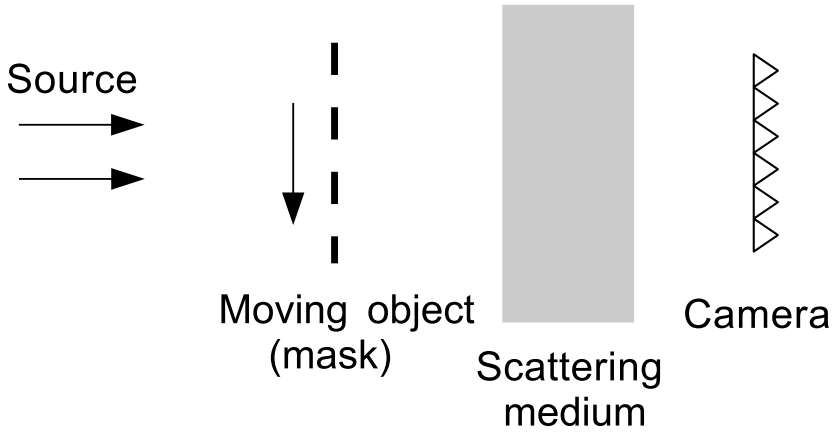
- The field in the plane of the camera (in the plane $z = L$) is denoted by $\hat{\phi}_{\boldsymbol{r}}(\boldsymbol{x})$.

- The measured intensity correlation is

$$\begin{aligned} \mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'} &= \frac{1}{|A_0|}\int_{A_0} |\hat{\phi}_{\boldsymbol{r}}(\boldsymbol{x})|^2 |\hat{\phi}_{\boldsymbol{r}'}(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &\quad -\Big(\frac{1}{|A_0|}\int_{A_0} |\hat{\phi}_{\boldsymbol{r}}(\boldsymbol{x})|^2 d\boldsymbol{x}\Big)\Big(\frac{1}{|A_0|}\int_{A_0} |\hat{\phi}_{\boldsymbol{r}'}(\boldsymbol{x})|^2 d\boldsymbol{x}\Big), \end{aligned}$$

where $A_0$ is the spatial support of the camera.

# Speckle intensity correlation imaging through a scattering medium

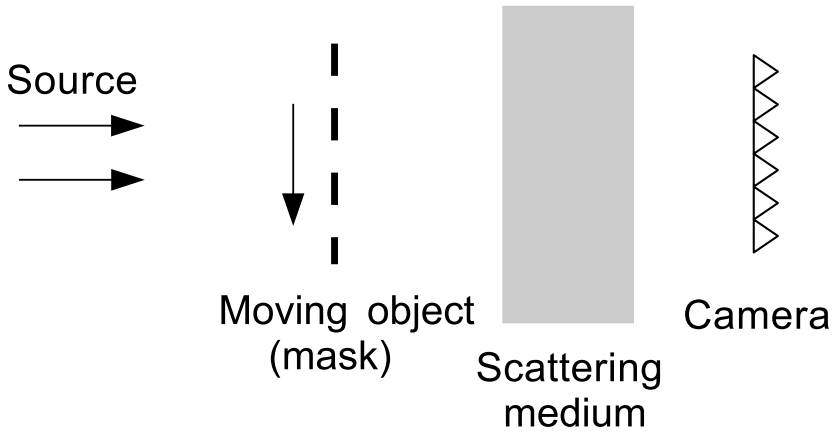



- Result (in the paraxial regime):

$$\begin{aligned}\mathbb{E}\big[\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'}\big] \;=\; & \int_{A_0} d\boldsymbol{X} \int d\boldsymbol{Y} \Big| \frac{1}{(2\pi)^2} \int \Big( \int f\big(\boldsymbol{x}+\frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big)\overline{f}\big(\boldsymbol{x}-\frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big) \exp\big(-i\boldsymbol{\zeta}\cdot\boldsymbol{x}\big) d\boldsymbol{x}\Big) \\ & \times \exp\Big(i\boldsymbol{\zeta}\cdot\big(\boldsymbol{X}-\frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\big)\Big) \exp\Big(\frac{\omega^2}{4c_o^2}\int_0^L \gamma\big(\frac{c_o\boldsymbol{\zeta}}{\omega}z-\boldsymbol{Y}\big)-\gamma(\boldsymbol{0})dz\Big) d\boldsymbol{\zeta}\Big|^2 \\ & - \Big| \frac{1}{(2\pi)^2} \int_{A_0} d\boldsymbol{X} \Big( \int f\big(\boldsymbol{x}+\frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big)\overline{f}\big(\boldsymbol{x}-\frac{\boldsymbol{r}'-\boldsymbol{r}}{2}\big) \exp\big(-i\boldsymbol{\zeta}\cdot\boldsymbol{x}\big) d\boldsymbol{x}\Big) \\ & \times \exp\Big(i\boldsymbol{\zeta}\cdot\big(\boldsymbol{X}-\frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\big)\Big) \exp\Big(-\frac{\omega^2}{4c_o^2}\gamma(\boldsymbol{0})L\Big) d\boldsymbol{\zeta}\Big|^2, \end{aligned}$$

with $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0},0)\mu(\boldsymbol{x},z)]dz$.

# Speckle intensity correlation imaging through a scattering medium




- Result: When $L \gg Z_{\text{sca}} := \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$ and

$$|A_0|(\sim \text{diam(camera)}^2) \gg \rho_L^2 := \frac{Z_{\text{sca}}\ell_c^2}{L}$$

($\rho_L$ = speckle radius), we have

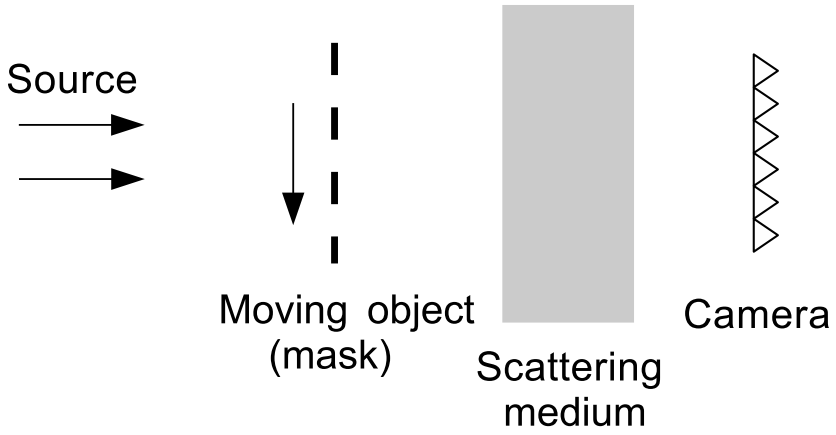
$$\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'} \simeq \mathbb{E}\big[\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'}\big] \approx \left| \int |\hat{f}(\boldsymbol{\kappa})|^2 \exp\big(i\boldsymbol{\kappa}\cdot(\boldsymbol{r}'-\boldsymbol{r})\big) d\boldsymbol{\kappa} \right|^2,$$

up to a multiplicative constant, where

$$\hat{f}(\boldsymbol{\kappa}) = \int f(\boldsymbol{x}) \exp\big(-i\boldsymbol{\kappa}\cdot\boldsymbol{x}\big) d\boldsymbol{x}.$$

$\hookrightarrow$ It is possible to reconstruct the mask indicator function $f$.

# Speckle intensity correlation imaging through a scattering medium




- We have

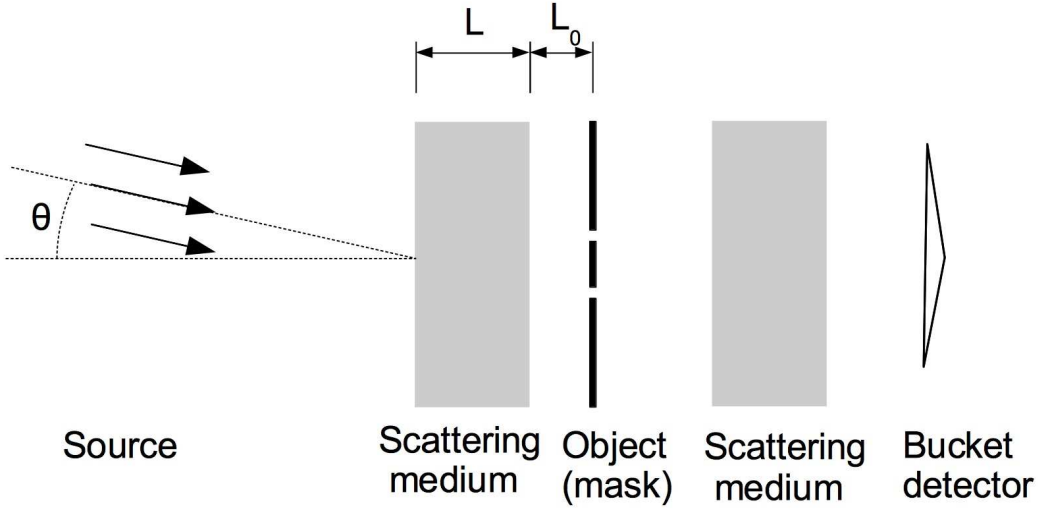
$$\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'} \simeq \mathbb{E}\big[\mathcal{C}_{\boldsymbol{r},\boldsymbol{r}'}\big] \approx \left| \int |\hat{f}(\boldsymbol{\kappa})|^2 \exp\big(i\boldsymbol{\kappa}\cdot(\boldsymbol{r}'-\boldsymbol{r})\big) d\boldsymbol{\kappa} \right|^2$$

$\hookrightarrow$ It is possible to reconstruct the incident field $f$ by a two-step phase retrieval algorithm (Gerchberg-Saxon-type).

1) Given $C_{\boldsymbol{r},\boldsymbol{r}'}$, we know the modulus of the (I)FT of $|\hat{f}(\boldsymbol{\kappa})|^2$, and we know the phase of $|\hat{f}(\boldsymbol{\kappa})|^2$ (zero) $\rightarrow$ we can extract $|\hat{f}(\boldsymbol{\kappa})|^2$.
2) Given $|\hat{f}(\boldsymbol{\kappa})|^2$, we know the modulus of the FT of $f(\boldsymbol{x})$, and we know the phase of $f(\boldsymbol{x})$ (zero) $\rightarrow$ we can extract $f(\boldsymbol{x})$.

$\rightarrow$ not very stable.

# Speckle intensity correlation imaging through a scattering medium (II)

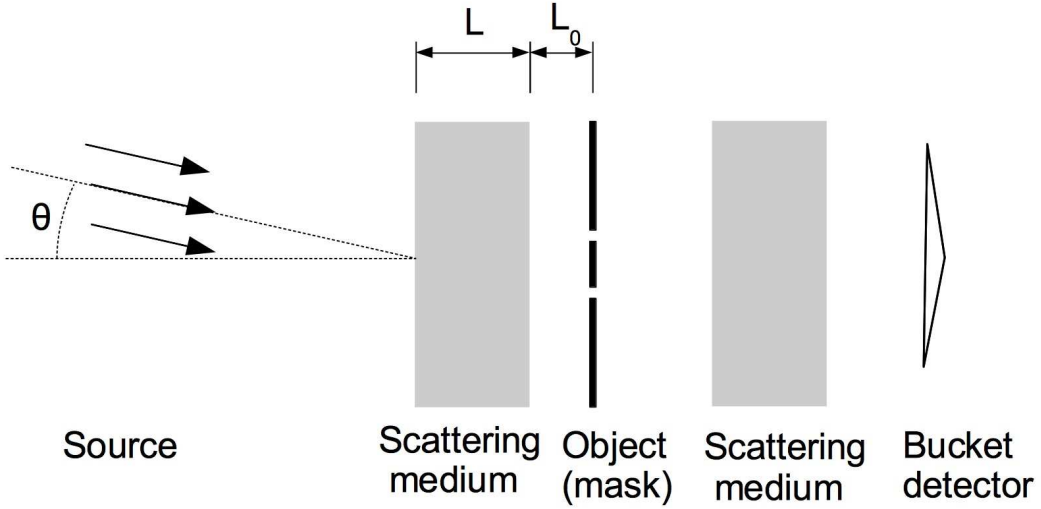



## Experimental set-up

- A laser beam with incident angle $\boldsymbol{\theta}$ is shined on the scattering medium.
- The object to be imaged is a mask.
- The total intensity of the light that goes through the mask is collected by a bucket detector.

$\rightarrow$ For each incident angle $\boldsymbol{\theta}$ the total transmitted intensity $\mathcal{E}_{\boldsymbol{\theta}}$ is measured.

# Speckle intensity correlation imaging through a scattering medium (II)




Consider:

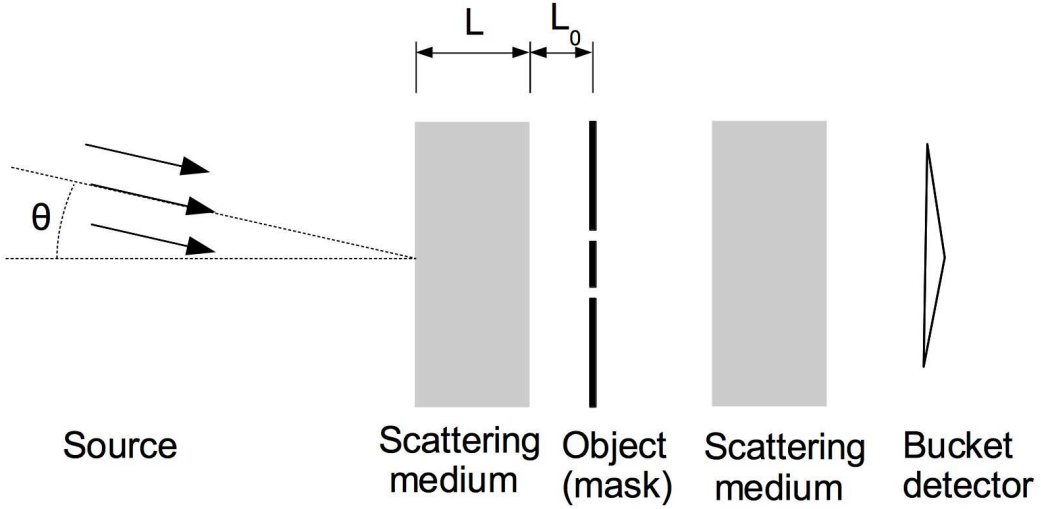
$$\mathcal{C}(\Delta\boldsymbol{\theta}) = \frac{1}{\Theta}\int_{\Theta} \mathcal{E}_{\boldsymbol{\theta}}\mathcal{E}_{\boldsymbol{\theta}+\Delta\boldsymbol{\theta}}d\boldsymbol{\theta} - \Big(\frac{1}{\Theta}\int_{\Theta}\mathcal{E}_{\boldsymbol{\theta}}d\boldsymbol{\theta}\Big)^2$$

- Result (in the paraxial regime):

$$\mathbb{E}[\mathcal{C}(\Delta\boldsymbol{\theta})] = \frac{1}{(2\pi)^2}\iint \exp\Big(\frac{\omega^2}{2c_o^2}\int_0^L \gamma\big(\boldsymbol{x}+\Delta\boldsymbol{\theta}(z+L_o)\big)dz\Big)e^{-i\boldsymbol{x}\cdot\boldsymbol{\kappa}}|\widehat{f^2}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa}d\boldsymbol{x}$$
$$\times \exp\Big(-\frac{\omega^2\gamma(\mathbf{0})L}{2c_o^2}\Big) - |\widehat{f^2}(\mathbf{0})|^2\exp\Big(-\frac{\omega^2\gamma_o(\mathbf{0})L}{2c_o^2}\Big),$$

with $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz$.

# Speckle intensity correlation imaging through a scattering medium (II)

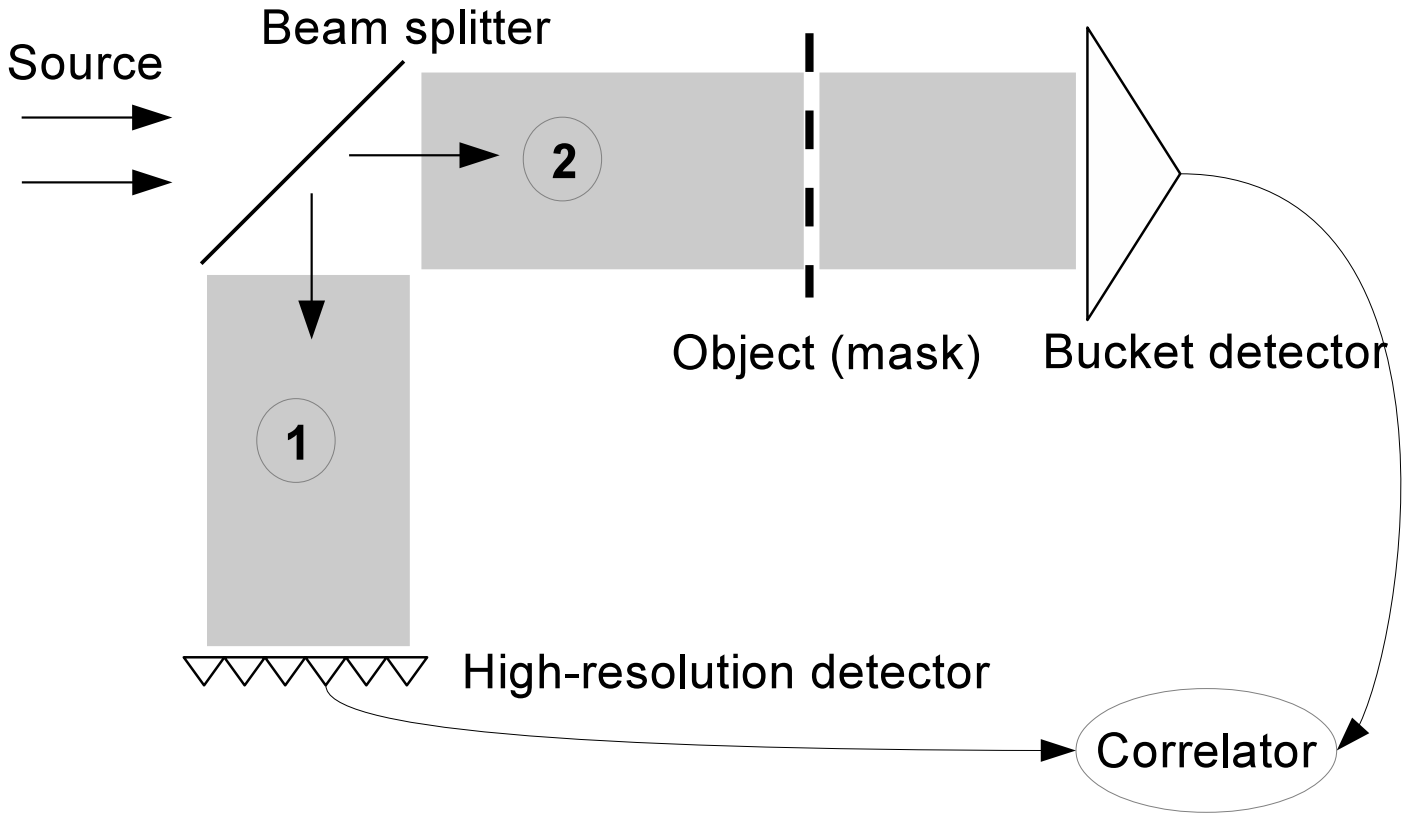



• Result: When $L \gg Z_{\rm sca} := \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$, $\rho_L^2 = \frac{Z_{\rm sca}\ell_c^2}{L}$ is small enough and $L_o$ is large enough, then

$$\mathcal{C}(\Delta\boldsymbol{\theta}) \simeq \mathbb{E}[\mathcal{C}(\Delta\boldsymbol{\theta})] \approx \int |\widehat{f^2}(\boldsymbol{\kappa})|^2 \exp\big(-i\boldsymbol{\kappa}\Delta\boldsymbol{\theta}L_o\big)d\boldsymbol{\kappa}$$

$\hookrightarrow$ One can extract $|\widehat{f^2}(\boldsymbol{\kappa})|^2$ (by Fourier transform or FFT) and then $f^2(\boldsymbol{x})$ by a (one-step) phase retrieval algorithm.

# Speckle intensity correlation imaging through a scattering medium (III)




- Noise source (laser light passed through a rotating glass diffuser).
- without object in path 1; a high-resolution detector measures the spatially-resolved intensity $I_1(t, \boldsymbol{x})$.
- with object (mask) in path 2; a single-pixel detector measures the spatially-integrated intensity $I_2(t)$.

Experiment: the correlation of $I_1(\cdot, \boldsymbol{x})$ and $I_2(\cdot)$ is an image of the mask [1,2].

[1] A. Valencia et al., *PRL* **94** (2005) 063601; [2] J. H. Shapiro et al., *Quantum Inf. Process* **1** (2012) 949.

# Speckle intensity correlation imaging through a scattering medium (III)

- Wave equation in paths 1 and 2:

$$\frac{1}{c_j^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u_j}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u_j = e^{-i\omega_o t} n(t,\boldsymbol{x})\delta(z) + c.c., \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \qquad j = 1, 2$$

- Noise source (with mean zero):

$$\Big\langle n(t,\boldsymbol{x})\overline{n(t,\boldsymbol{x}')}\Big\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_o^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}')$$

- Wave fields: $u_j(t,\vec{\boldsymbol{x}}) = v_j(t,\vec{\boldsymbol{x}})e^{-i\omega_o t} + c.c., \qquad j = 1, 2$
- Intensity measurements:

$$\begin{aligned} I_1(t,\boldsymbol{x}) &= |v_1(t,(\boldsymbol{x},L))|^2 \text{ in the plane of the high-resolution detector} \\ I_2(t) &= \int_{\mathbb{R}^2} |v_2(t,(\boldsymbol{x}',L+L_0))|^2 d\boldsymbol{x}' \text{ in the plane of the bucket detector} \end{aligned}$$

- Correlation:

$$C_T(\boldsymbol{x}) = \frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})I_2(t)dt - \Big(\frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})dt\Big)\Big(\frac{1}{T}\int_0^T I_2(t)dt\Big)$$

## Speckle intensity correlation imaging through a scattering medium (III)

- If the propagation distance is larger than the scattering mean free path, then

$$C_T(\boldsymbol{x}) \overset{T\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})^4 d\boldsymbol{y},$$

where $f(\boldsymbol{x})$ is the mask indicator function and $\mathcal{H}(\boldsymbol{x})$ is a convolution kernel [1].

- If the medium is homogeneous:

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4}{2^8\pi^2 L^4} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi0}^2}\Big), \qquad \rho_{\rm gi0}^2 = \frac{c_o^2 L^2}{2\omega_o^2 r_o^2}.$$

- If the medium in path 1 and 2 is random (*independent* realizations):

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4 \rho_{\rm gi0}^2}{2^8\pi^2 L^4 \rho_{\rm gi1}^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi1}^2}\Big), \qquad \rho_{\rm gi1}^2 = \rho_{\rm gi0}^2 + \frac{4c_o^2 L^3}{3\omega_o^2 Z_{\rm sca}\ell_{\rm c}^2}.$$

↪ Scattering (only slightly) reduces the resolution.

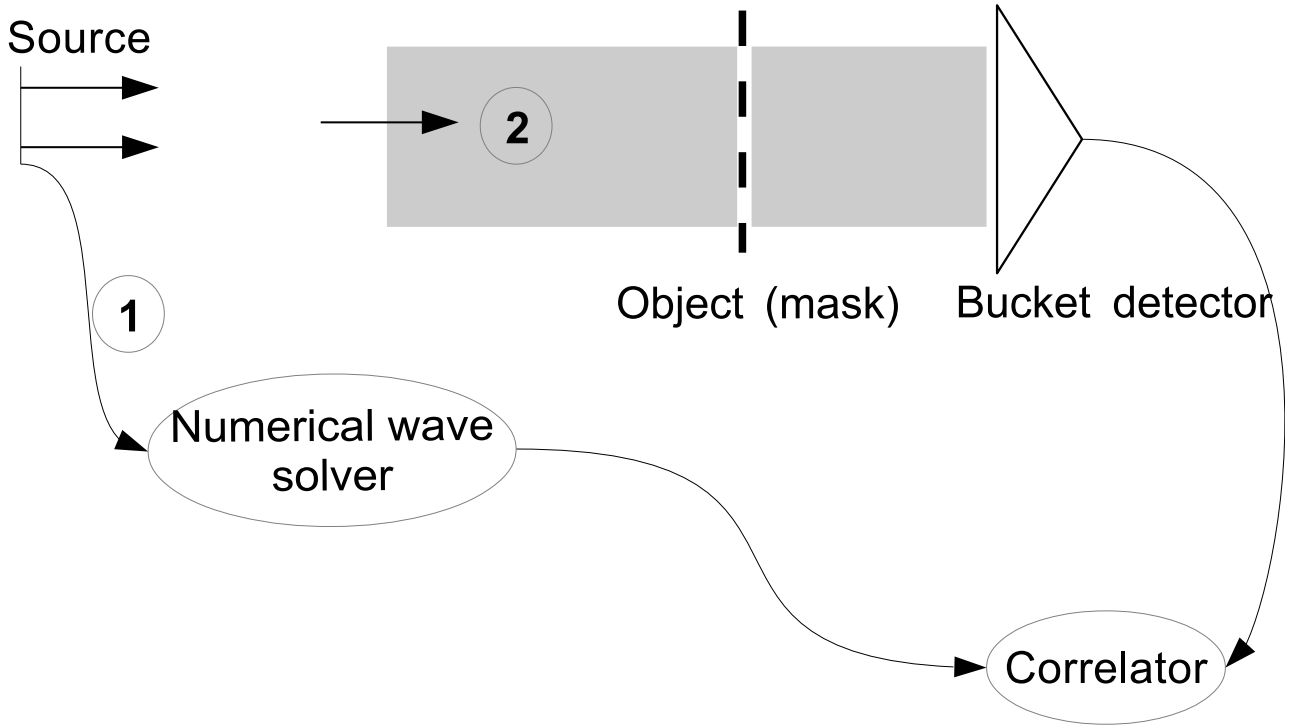
- If the medium in path 1 and 2 is random (*same* realization):

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4}{2^8\pi^2 L^4} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi2}^2}\Big), \qquad \frac{1}{\rho_{\rm gi2}^2} = \frac{1}{\rho_{\rm gi0}^2} + \frac{16L}{Z_{\rm sca}\ell_{\rm c}^2}.$$

↪ Scattering enhances the resolution !

[1] J. Garnier, *Inverse Problems and Imaging* **10** (2016) 409.

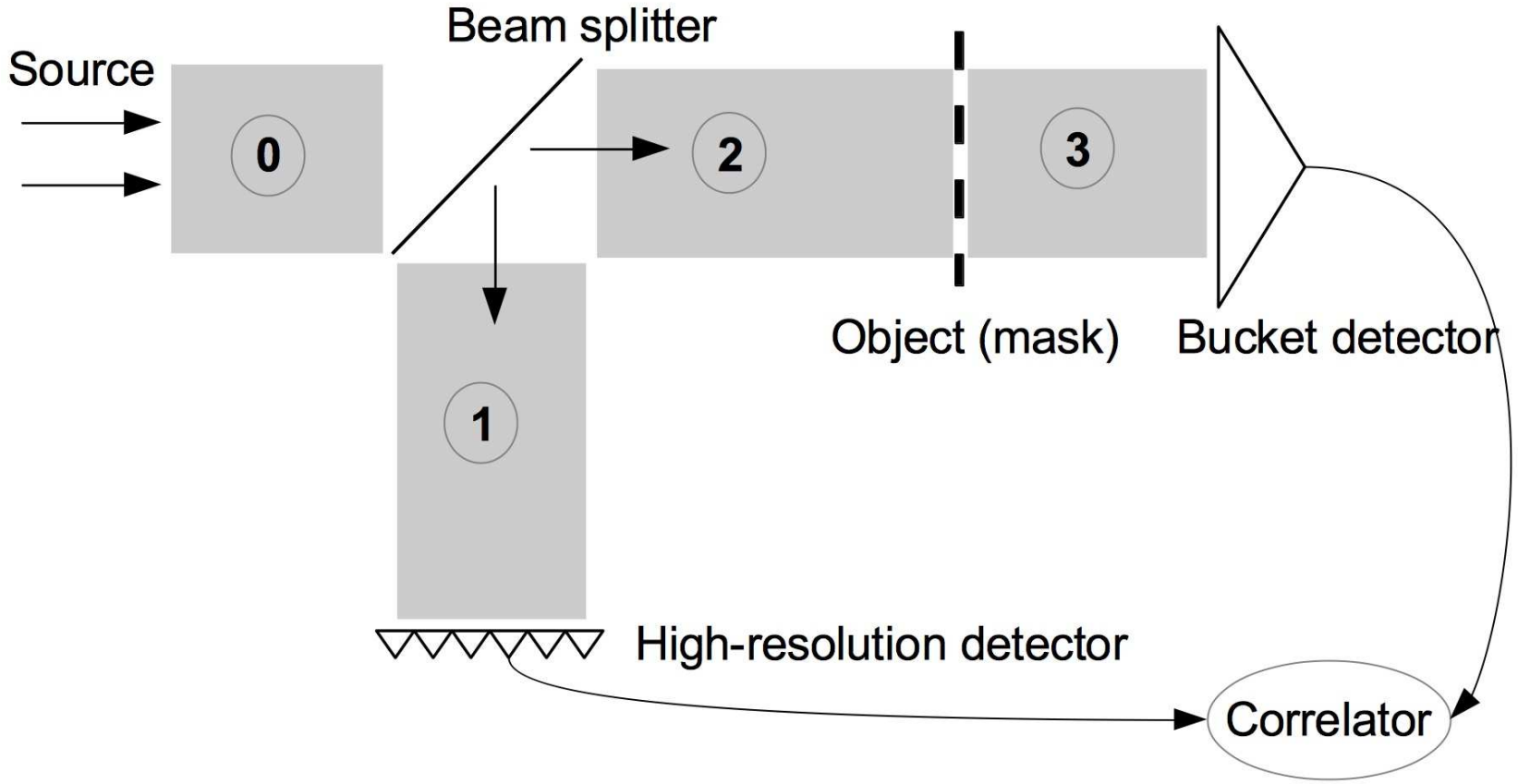
# Speckle intensity correlation imaging through a scattering medium (IV)




- The medium in path 2 is randomly heterogeneous.
- There is no other measurement than $I_2(t)$.
- The realization of the source is known (use of a Spatial Light Modulator) and the medium is taken to be homogeneous in the "virtual path 1" $\rightarrow$ one can *compute* the field (and therefore its intensity $I_1(t, \boldsymbol{x})$) in the "virtual" output plane of path 1.

$\hookrightarrow$ a *one-pixel camera* can give a high-resolution image of the mask!

# Final remark: On the role of the random medium




*Is random medium good or bad for imaging ?*

Random medium in region 0 is *good*.

Random medium in regions 1 and 2 is *bad*.

Random medium in region 3 plays *no role*.

## Conclusion

- Correlation-based imaging allows for imaging in randomly scattering media.
- One needs to process *well-chosen* cross correlations of the data.

- Fourth-order moment of the wave field is useful.
- First application: Scintillation index and stability of Wigner transform.
- Second application: Intensity correlation-based imaging, ghost imaging.