Increasing stability in the inverse source problem with attenuation and many frequencies

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Joint work with Gang Bao, Jin Cheng, Victor Isakov and William Rundell



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Outline

1. Introduction
   - Related work
   - Non-radiating source
   - Helmholtz equation
   - Main result
2. Increasing stability with attenuation
   - Analytic functions
   - Exact observability bounds
   - Proof of the main theorem
3. Numerical algorithm for source identification
   - Regularization method
   - Numerical examples
4. Ongoing works

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Related work

Multifrequency (highfrequency) regimes

Inverse medium problems Chen & Rokhlin (97); Chen (97a, 97b); Bao, Chen & Ma(00); Bao & Liu (03), Bao & Li (07); Bao & Triki (10); Nagayasu, Uhlmann & Wang (13); Bao & Triki (19)...

Inverse obstacle/interface problems Coifman, Goldberg, Hrycak & Rokhlin (99); Bao, Hou & Li (07); Bao & Lin (10, 11); Sini & Thành (12); Borges & Greengard (15); Rondi & Sini (15)...

Inverse source problems Bao, Lin & Triki (10, 11); Bao, Lu, Rundell & Xu; Cheng, Isakov & Lu (16); Li & Yuan (17); & Isakov & Lu (18)...



Bao, Gang; Li, Peijun; Lin, Junshan; Triki, Faouzi *Inverse scattering problems with multi-frequencies*. Inverse Problems 31 (2015), no. 9, 093001, 21 pp.

Related work

Multifrequency (highfrequency) regimes

Continuation problems Subbarayappa & Isakov (07, 10); Isakov & Kindermann (11); Isakov (14)...

Schrödinger problems Isakov (11); Isakov, Nagayasu, Uhlmann & Wang (14); Isakov & Wang (14); Isakov, Lai & Wang (16); Isakov, Lu & Xu (19)...

Multifrequency electric impedance tomography Ammari & Triki (17); Cheng, Choulli & Lu (19)...

Types of measurements

Frequency sweep:

Inverse medium problems; inverse obstacle/interface problems;
inverse source problems; multifrequency electric impedance tomography

Dirichlet-to-Neumann map at a fixed high wavenumber

Inverse medium problems; Schrödinger problems

Cauchy data at a fixed high wavenumber

Continuation problems

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Continuation problems

Forward source problems

Helmholtz equation with inner source terms

Radiated field $u(x;k)$ satisfies Helmholtz equation and Sommerfeld radiation condition

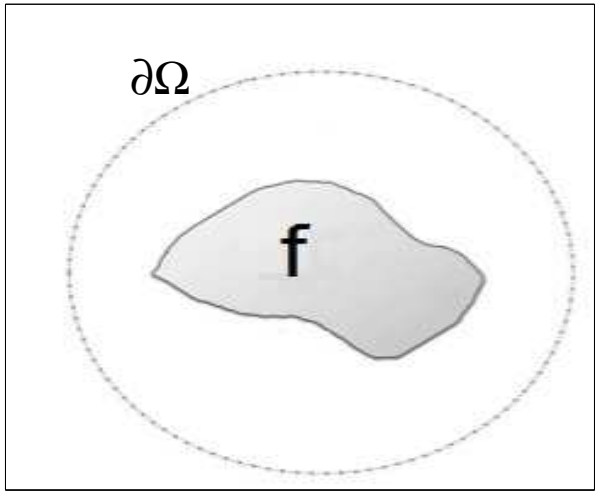
$$\Delta u(x;k) + k^2 u(x;k) = -f(x), \quad x \in \mathbb{R}^d$$

$$\lim_{r_x \to \infty} r_x^{\frac{d-1}{2}} \left(\frac{\partial u(x,k)}{\partial r_x} - \mathbf{i}ku(x,k) \right) = 0$$

where $k \geqslant 0$ is the wavenumber. We assume $f(x) \in L^2(\mathbb{R}^d)$ and compactly supported in a bounded open domain Ω.

Outgoing radiated field $u(x;k)$ is generated by the source function $f(x)$ and measured at $\partial\Omega$.

Forward map



Definition: Forward operator $\mathscr{L}_k$

If we assume $f^\dagger(x) \in L^2(\Omega)$ satisfying supp$f^\dagger \subset\subset \Omega$,

$$\begin{aligned}(\mathscr{L}_k f^\dagger)(x) &= u(x;k)|_{\partial\Omega} = g(x;k) \\ &= \int_\Omega f^\dagger(y)\Phi(x,y;k)\mathrm{d}y, \quad x \in \partial\Omega\end{aligned}$$

where $\Phi(x,y;k)$ is the fundamental solution. $\mathscr{L}_k : L^2(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$.

Non-radiating source

Non-radiating source at a wavenumber k

The outgoing radiated field $u(x;k) \in H^1(\Omega)$ corresponding to the non-zero source $f(x) \in L^2(\Omega)$ satisfies $u(x;k)|_{\partial\Omega} = 0$. We call $f(x)$ the non-radiating source for the wavenumber k.

For instance

$$\varnothing \neq \left\{ f(x) \in L^2(\Omega) \mid f(x) = -\Delta w(x) - k^2 w(x),\ w(x) \in C_0^\infty(\Omega) \right\}.$$

Bleistein and Cohen [J. Math. Phy. 1977]; Kim and Wolf [Optics Comm. 1986]; Marengo and Ziolkowski [PRL 1999].

Spherical Fourier transform

Spherical Fourier basis

Let $\Omega = B_\rho$ in $\mathbb{R}^3$ with $\rho > 0$. Choose

$$\psi_{nm\ell}(x) = (N_{n\ell})^{-1/2} j_n(\zeta_{n\ell} r_x) Y_n^m(\hat{x})$$

with $\zeta_{n\ell} = \frac{z_{nl}}{\rho}$, $\ell = 1, 2, \ldots$, $z_{n\ell}$ is the ℓ-th positive zero of the spherical Bessel function $j_n(r)$, $N_{n\ell} = \frac{\rho^3}{2}\left(\frac{d}{dr} j_n(\zeta_{n\ell}\rho)\right)^2$, $r_x = |x|$ and $\hat{x} = \frac{x}{|x|}$.

Spherical Fourier transform

For all $f \in L^2(\Omega)$,

$$a_{nm\ell} = \int_\Omega f(y)\overline{\psi_{nm\ell}(y)}dy.$$

Spherical Fourier transform

Notice the fact that

$$f(x) = \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \sum_{m=-n}^{n} a_{nm\ell} \psi_{nm\ell}(x),$$

$$\Phi(x,y;k) = \frac{e^{\mathbf{i}k|x-y|}}{4\pi|x-y|} = \mathbf{i}k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(kr_x) Y_n^m(\hat{x}) j_n(kr_y) \overline{Y_n^m(\hat{y})}.$$

by choosing $k = \zeta_{n\ell}$, we obtain

$$\begin{aligned}
(\mathscr{L}_k \psi_{n'm'\ell'})(x) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathbf{i}k h_n^{(1)}(kr_x) Y_n^m(\hat{x}) \left(\int_{\Omega} \psi_{n'm'\ell'}(y) j_n(kr_y) \overline{Y_n^m(\hat{y})} \mathrm{d}y \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathbf{i}k h_n^{(1)}(kr_x) Y_n^m(\hat{x}) \left(N_{n'\ell'}^S \right)^{\frac{1}{2}} \delta_{n=n',m=m'} \delta_{k=\zeta_{n'\ell'}} \\
&= \mathbf{i}k h_{n'}^{(1)}(kr_x) Y_{n'}^{m'}(\hat{x}) \left(N_{n'\ell'}^S \right)^{\frac{1}{2}} \delta_{k=\zeta_{n'\ell'}}, \quad x \in \partial\Omega.
\end{aligned}$$

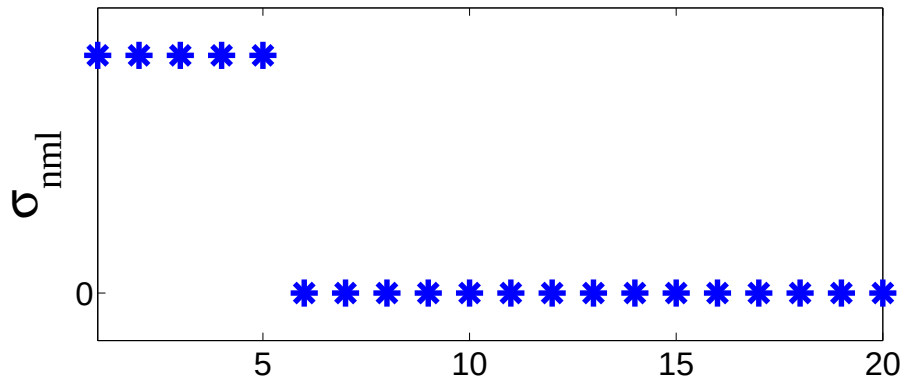
Singular value decomposition (SVD)

SVD of the forward operator $\mathscr{L}_k$

Assume $\Omega = B_\rho$ in $\mathbb{R}^3$, the singular value system of the forward operator $\mathscr{L}_k : L^2(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is $\{\sigma_{nm\ell}, \psi_{nm\ell}, \phi_{nm\ell}\}$ when $k = \zeta_{n\ell}$.

Here $\psi_{nm\ell}(x) = (N_{n\ell})^{-1/2} j_n(\zeta_{n\ell} r_x) Y_n^m(\hat{x})$ with $x \in \Omega$. Furthermore, for all $\ell = 1, 2, \cdots$, $\phi_{nm\ell}(x) = Y_n^m(\hat{x})$ with $x = \rho\hat{x}$, and for all $m = 0, \pm 1, \cdots, \pm n$, $n = 0, 1, \cdots$, the singular values $\sigma_{nm\ell}$ are

$$\sigma_{nm\ell}(k) = \left| \mathbf{i} k h_n^{(1)}(k\rho) \right| (N_{n\ell})^{\frac{1}{2}} \delta_{k=\zeta_{n\ell}} \geqslant 0.$$



Inverse source problems

Ideal inverse source problems

Identify the source function $f(x)$ from the measured data (Dirichlet, Neumann, Cauchy) on the boundary $\partial\Omega$ for wavenumbers $k \in (0, +\infty)$.

Real situation

Identify the source function $f(x)$ from the measured data (Dirichlet, Neumann, Cauchy) on the boundary $\partial\Omega$ for wavenumbers $k \in (0, K)$, $K >> 0$.



Hoenders and Ferwerda [PRL 2001]; Devaney et. al [SIAP 2007]; Bao, Lin and Triki [JDE 2010].

Inverse source problems

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Helmholtz equation

Helmholtz equation with attenuation

Radiated field $u(x;k)$ satisfies Helmholtz equation

$$(\Delta + k^2 + \mathbf{i}kb)u = -f_1 - bf_0 + \mathbf{i}kf_0 \quad \text{in}\,\mathbb{R}^3,\ 0 < k < K,$$

with the exponential decay at infinity: $|u(x)| + |\nabla u(x)| \leq C(u)e^{-\delta(u)|x|}$ and some $\delta(u) > 0$.

We assume $f_0, f_1 \in H^3(\mathbb{R}^3)$ and compactly supported in a bounded open domain Ω.
For special case of $b = 0$, we refer to



[J. Cheng, V. Isakov and S. Lu, 16]

Main theorem

There exists a generic constant C only depending on the domain Ω such that

$$\|f_0\|^2_{(1)}(\Omega) + \|f_1\|^2_{(0)}(\Omega) \leq Ce^{Cb^2}\left(\varepsilon^2 + \frac{M_3^2}{1+K^{\frac{4}{3}}|E|^{\frac{1}{2}}}\right)$$

for all $u \in H^2(\Omega)$ solving the above Helmholtz equation, with $1 < K$. Here

$$\varepsilon^2 = \int_0^K \left(\omega^2 \|u(,\omega)\|^2_{(0)}(\partial\Omega) + \|\nabla u(,\omega)\|^2_{(0)}(\partial\Omega)\right) d\omega,$$

$E = -\ln\varepsilon$ and $M_3 = \max\left\{\|f_0\|_{(3)}(\Omega) + \|f_1\|_{(3)}(\Omega), 1\right\}$ where $\|\cdot\|_{(\ell)}(\Omega)$ is the standard Sobolev norm in $H^\ell(\Omega)$.

 [V. Isakov and S. Lu, 18]

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Solution of the Helmholtz equation

$$u^*(x,\kappa) := u(x,k),\ \kappa := k\sqrt{1+\frac{b}{k}\mathbf{i}},$$

then the Helmholtz equation becomes

$$\Delta u^* + \kappa^2 u^* = -f_1 - bf_0 + \mathbf{i}kf_0$$

with a solution

$$u^*(x,\kappa) = \frac{1}{4\pi}\int_\Omega (f_1 + bf_0 - \mathbf{i}kf_0)(y)\frac{e^{\mathbf{i}\kappa|x-y|}}{|x-y|}dy$$

which exponentially decays for large $|x|$ because the imaginary part $\mathrm{Im}(\kappa) > 0$.

Frequency domain $\Longrightarrow$ Time domain

Wave equation

Wave equation

The hyperbolic initial value problem

$$\partial_t^2 U - \Delta U + b\partial_t U = 0 \text{ on } \Omega \times (0, +\infty),$$
$$U(x,0) = f_0,\ \partial_t U(x,0) = f_1 \text{ on } \Omega.$$

Fourier transform

$$u(x,k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} U(x,t) e^{\mathbf{i}kt} dt.$$

Why time domain?

Exact observability bound

Exact observability bound

Let the observation time $4(D+1) < T < 5(D+1)$. Then there exists a generic constant C depending on the domain Ω such that

$$\|f_0\|^2_{(1)}(\Omega) + \|f_1\|^2_{(0)}(\Omega) \leq Ce^{Cb^2}\left(\|\partial_t U\|^2_{(0)}(\partial\Omega \times (0,T)) + \|\nabla U\|^2_{(0)}(\partial\Omega \times (0,T))\right)$$

for all $U \in H^2(\Omega \times (0,+\infty))$ solving the hyperbolic initial problem.

Proof based on the Carleman estimate.



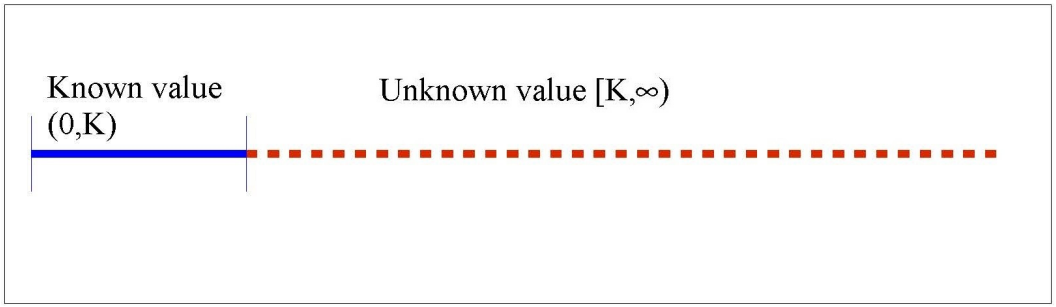
Klibanov & Malinsky 1991; Tataru 1995.

The main mathematical problem!

Equivalency of the boundary data

$$\|\partial_t U\|^2_{(0)}(\partial\Omega\times(0,T)) = 2\int_0^\infty \omega^2 \int_{\partial\Omega} u(\cdot,\omega)u(\cdot,-\omega)d\Gamma d\omega;$$

$$\|\nabla U\|^2_{(0)}(\partial\Omega\times(0,T)) = 2\int_0^\infty \int_{\partial\Omega} \nabla u(\cdot,\omega)\cdot\nabla u(\cdot,-\omega)d\Gamma d\omega.$$



Analytic functions

Measurement

$$\varepsilon^2 = I_0(K) + I_1(K).$$

Analytic functions

$$I_0(k) := 2\int_0^k \omega^2 \int_{\partial\Omega} u(\cdot,\omega)u(\cdot,-\omega)d\Gamma d\omega,$$

$$I_1(k) := 2\int_0^k \int_{\partial\Omega} \nabla u(\cdot,\omega)\cdot\nabla u(\cdot,-\omega)d\Gamma d\omega,$$

which are analytic functions with respect to the wave number $k \in \mathbb{C}\setminus[-b\mathbf{i},0]$.

Harmonic measure

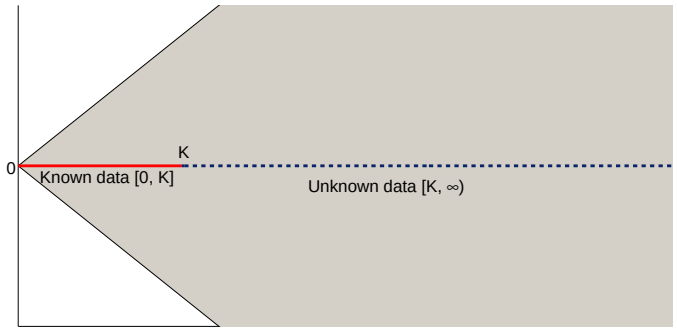
Boundness of both values

Let $\mathrm{supp} f_0, \mathrm{supp} f_1 \subset \Omega$ and $f_1 \in H^1(\Omega), f_0 \in H^1(\Omega)$. $\mathbb{S}$ is the sector $\left\{|\arg k| < \frac{\pi}{4}\right\}$ or $\{|k_2| < k_1\}$ with $k = k_1 + k_2\mathbf{i}$. Then, in $\mathbb{S}$, following bounds holds true

$$|I_0(k)| \leq 24\pi|\partial\Omega|D\left(\frac{1}{3}|k|^3\|f_1\|^2_{(0)}(\Omega) + \left(\frac{1}{3}|k|^3b^2 + \frac{1}{5}|k|^5\right)\|f_0\|^2_{(0)}(\Omega)\right)e^{D(4k_1+b)},$$

$$|I_1(k)| \leq 24\pi|\partial\Omega|D\left(|k|\|f_1\|^2_{(1)}(\Omega) + \left(|k|b^2 + \frac{1}{3}|k|^3\right)\|f_0\|^2_{(1)}(\Omega)\right)e^{D(4k_1+b)},$$

where $|\partial\Omega|$ is the area of $\partial\Omega$ and $D = \sup|x-y|$ over $x, y \in \Omega$.



Harmonic measure

$$\begin{aligned}
&\left|I_0(k)e^{-(4D+1)k}\right| \\
&\quad \le 24\pi|\partial\Omega|De^{Db}\left(\frac{2\sqrt{2}}{3}k_1^3\|f_1\|^2_{(0)}(\Omega)+\left(\frac{2\sqrt{2}}{3}k_1^2b^2+\frac{4\sqrt{2}}{5}k_1^5\right)\|f_0\|^2_{(0)}(\Omega)\right)e^{-k_1} \\
&\quad \le Cb^2e^{Db}M_0^2,
\end{aligned}$$

with $M_0 = \max\left\{\|f_0\|_{(0)}(\Omega)+\|f_1\|_{(0)}(\Omega),1\right\}$. Noticing that

$$\left|I_0(k)e^{-(4D+1)k}\right| \le \varepsilon^2 \text{ on } [0,K],$$

we conclude that

$$\left|I_0(k)e^{-(4D+1)k}\right| \le Cb^2e^{Db}\varepsilon^{2\mu(k)}M_0^2,$$

when $K < k < +\infty$

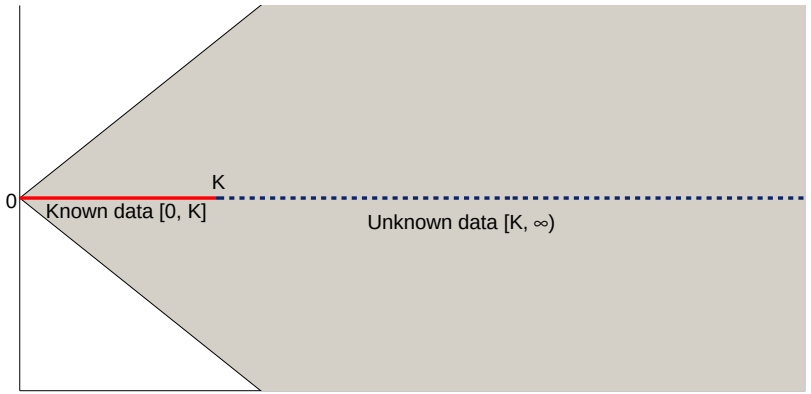
Harmonic measure

Lower bound of the Harmonic measure

Let $\mathbb{S}$ be the sector $\{k : -\frac{\pi}{4} < \arg k < \frac{\pi}{4}\}$ and $\mu(k)$ be the harmonic measure of the interval $[0, K]$ in $\mathbb{S} \setminus [0, K]$.

If $0 < k < 2^{\frac{1}{4}} K$, $\mu(k) > \frac{1}{2}$.

If $2^{\frac{1}{4}} K < k$, $\mu(k) > \frac{1}{\pi} \left(\left(\frac{k}{K} \right)^4 - 1 \right)^{-\frac{1}{2}}$.



[J. Cheng, V. Isakov and S. Lu, 16]

Auxiliary equalities

Auxiliary equalities

$$\int_{-\infty}^{\infty} \omega^2 \|u(\cdot,\omega)\|^2_{(0)}(\partial\Omega)d\omega = I_0(k) + \int_{k<|\omega|} \omega^2 \|u(\cdot,\omega)\|^2_{(0)}(\partial\Omega)d\omega,$$

$$\int_{-\infty}^{\infty} \|\nabla u(\cdot,\omega)\|^2_{(0)}(\partial\Omega)d\omega = I_1(k) + \int_{k<|\omega|} \|\nabla u(\cdot,\omega)\|^2_{(0)}(\partial\Omega)d\omega.$$

Auxiliary lemma

Boundness of the high frequency modes

Let u be a solution to the Helmholtz equation with $f_1 \in H^3(\Omega)$ and $f_0 \in H^3(\Omega)$, $\text{supp} f_0, \text{supp} f_1 \subset \Omega$. Then

$$\int_{k<|\omega|} \omega^2 \|u(\cdot,\omega)\|^2_{(0)}(\partial\Omega)d\omega + \int_{k<|\omega|} \|\nabla u(\cdot,\omega)\|^2_{(0)}(\partial\Omega)$$
$$\leq Cb^5 k^{-2}\left(\|f_0\|^2_{(3)}(\Omega) + \|f_1\|^2_{(3)}(\Omega)\right).$$

Proof based on the decay property of the Cauchy problem for a semi-linear wave equation.

Kawashima et. al 1995.

Increasing stability

Increasing stability theorem

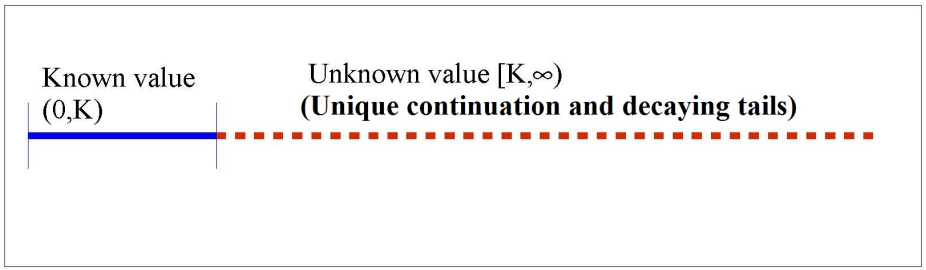
There exists a generic constant C only depending on the domain Ω such that

$$\|f_0\|^2_{(1)}(\Omega) + \|f_1\|^2_{(0)}(\Omega) \leq C e^{Cb^2} \left(\varepsilon^2 + \frac{M_3^2}{1 + K^{\frac{4}{3}} |E|^{\frac{1}{2}}} \right)$$

for all $u \in H^2(\Omega)$ solving the above Helmholtz equation, with $1 < K$. Here

$$\varepsilon^2 = \int_0^K \left(\omega^2 \|u(,\omega)\|^2_{(0)}(\partial\Omega) + \|\nabla u(,\omega)\|^2_{(0)}(\partial\Omega) \right) d\omega,$$

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Regularization method

Radiated field $u(x;k)$ satisfies Helmholtz equation and Sommerfeld radiation condition

$$(\Delta + k^2 + \mathbf{i}kb)u = -f \quad \text{in}\,\mathbb{R}^3,$$

with $f \in L^2(\Omega)$ compactly supported in Ω.

Inverse source problems

Identify the source f from the near field data

$$u = u_0,\ \partial_\nu u = u_1 \ \text{ on } \ \partial\Omega, \ \text{with } 1 < k < K.$$

Regularization method

Algorithm 1: Recursive Kaczmarz-Landweber Iteration

Input: Wave number set $\mathscr{K}$ with J discrete wave numbers $k_j \in [1,K]$, $j = \{1,2,\ldots,J\}$; measurement data $\left(u_{\kappa_j}, \frac{\partial u_{\kappa_j}}{\partial \nu}\right)|_\Gamma$ for all $\kappa_j = \sqrt{k_j^2 + \mathbf{i}k_j b}$ with $k_j \in \mathscr{K}$.

Output: Reconstructed source function f_{2N}^J with J & $2N$ denoting the Kaczmarz & Landweber iteration number.

1: Set the initial guess $f_{2N}^0 = 0, j, n = 0$.
2: **For** $j = 1,2,\ldots,J$ (Kaczmarz iteration)
3: $k = k_j; f_0^j = f_{2N}^{j-1}$;
4: **For n=1,2,...N** (Landweber iteration for Dirichlet & Neumann measurement)
5: $f_n^j = f_{n-1}^j + \mu(L_\kappa^1)^*(u_\kappa - L_\kappa^1 f_{n-1}^j), \qquad f_n^j = f_{n-1}^j + \mu(L_\kappa^2)^*\left(\frac{\partial u_\kappa}{\partial \nu} - L_\kappa^2 f_{n-1}^j\right)$;
6: $n = n+1$;
7: **End**
8: **End**.

[G. Bao, S. Lu, W. Rundell and B. Xu, 15]

Convergence

Convergence theorem

Let the exact source function $f^\dagger \in H_0^\tau(\Omega)$ with $\tau > \frac{d}{2}$. Then for a sufficiently large wavenumber interval, given any $\varepsilon > 0$, there exist finite iterations such that the final iterate in the recursive algorithm approximates the exact solution

$$\|f_j^i - f^\dagger\| \leq \varepsilon.$$

Error estimate

Error estimate theorem

Let the exact source function $f^\dagger \in H_0^\tau(\Omega)$ with $\tau > \frac{d}{2}$ and δ be the noise level at each wavenumber. The stopping criteria for the recursive algorithm is well-defined as the smallest integer j_{DP} with $\|\mathscr{L}_k f_{j_{DP}}^{i,\delta} - g_k^\delta\| \leq r\delta$, $r > 1$. Then following error estimate holds true

$$\|f_{j_{DP}}^{i,\delta} - f^\dagger\| \leq C\delta^{\frac{\tau}{\tau+1}} + \varepsilon \quad \text{for near field datum.}$$

Exact Solution

$$
\begin{aligned}
f(x,y,z) =& 1.1e^{-200((y-0.12)^2+(x-0.01)^2+(z-0.00)^2)} + 0.5e^{-180((y-0.01)^2+(x+0.01)^2+(z-0.00)^2)} \\
&+ 2e^{-180((y-0.21)^2+(x-0.11)^2+(z-0.00)^2)} + 0.8e^{-180((y+0.05)^2+(x+0.05)^2+(z-0.00)^2)} \\
&+ 0.6e^{-160((y-0.12)^2+(x-0.04)^2+(z-0.12)^2)} + 0.7e^{-170((y-0.02)^2+(x-0.06)^2+(z-0.07)^2)} \\
&+ 0.8e^{-160((y+0.12)^2+(x+0.04)^2+(z+0.12)^2)} + 0.7e^{-190((y+0.02)^2+(x+0.06)^2+(z+0.07)^2)} \\
&+ 1e^{-180((y-0.06)^2+(x-0.08)^2+(z-0.15)^2)} + 1.1e^{-190((y-0.09)^2+(x-0.1)^2+(z-0.03)^2)} \\
&+ 0.6e^{-200((y+0.06)^2+(x+0.08)^2+(z+0.15)^2)} + 1.3e^{-210((y+0.09)^2+(x+0.1)^2+(z+0.03)^2)} \\
&+ 1.2e^{-220((y-0.13)^2+(x+0.02)^2+(z+0.20)^2)} + 0.5e^{-230((y+0.02)^2+(x-0.08)^2+(z-0.2)^2)} \\
&+ 0.7e^{-150((y-0.21)^2+(x-0.04)^2+(z-0.03)^2)} + 0.9e^{-200((y-0.05)^2+(x+0.06)^2+(z+0.09)^2)} \\
&+ 0.9e^{-250((y+0.02)^2+(x+0.16)^2+(z+0.14)^2)} + 1.5e^{-240((y-0.01)^2+(x-0.11)^2+(z-0.14)^2)}.
\end{aligned}
$$

Approximate Solution: $K = 200$ and exact measurement

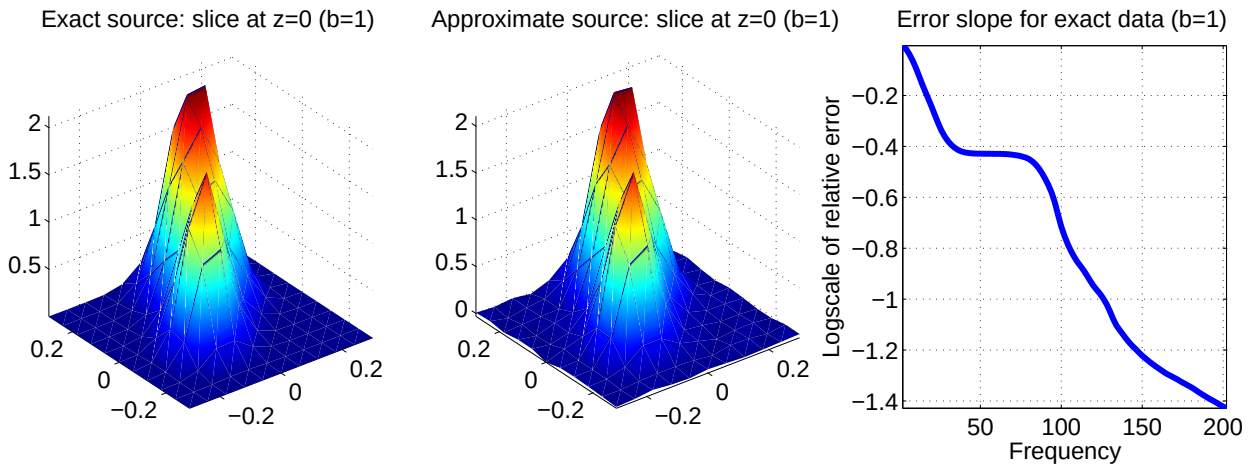


Figure: ($b = 1$) Left: exact source: slide of $z = 0$. Middle: approximate source: slide of $z = 0$ for exact data. Right: error slope for relative error.

Approximate Solution: $K = 200$ and noisy measurement

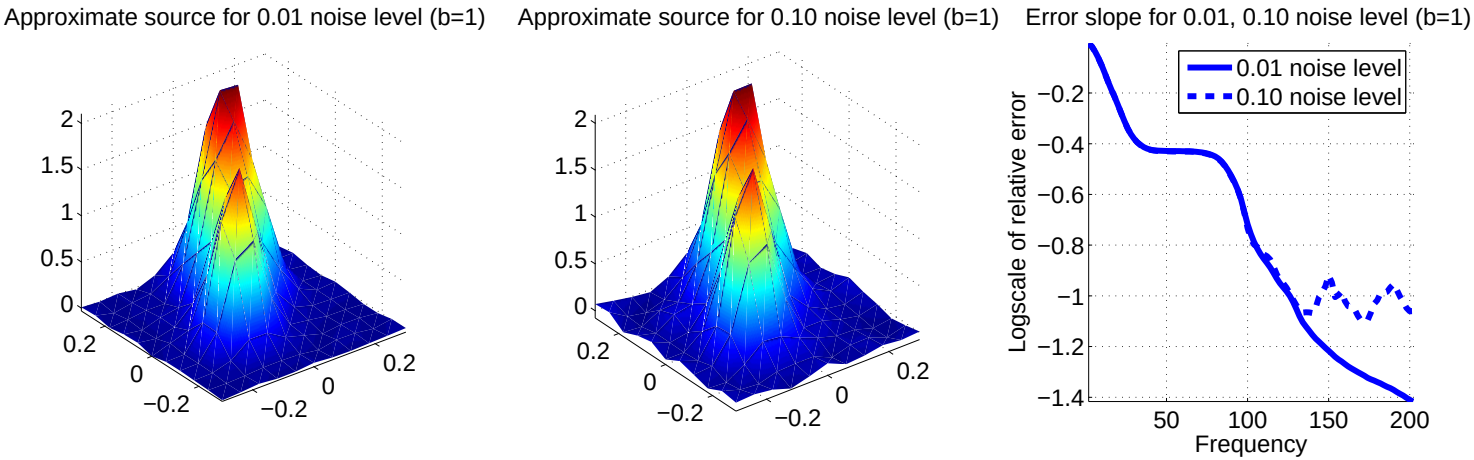


Figure: ($b = 1$) Left: approximate source: slide of $z = 0$ for 0.01 noise level. Middle: approximate source: slide of $z = 0$ for 0.1 noise level. Right: error slope for relative error for both noises.

Different choices of b

b value	Relative error for exact data
$b = 0.1$	0.0307
$b = 1$	0.0373
$b = 5$	0.0974
$b = 10$	0.2451
$b = 15$	0.5866
$b = 20$	0.9118
$b = 30$	0.9492

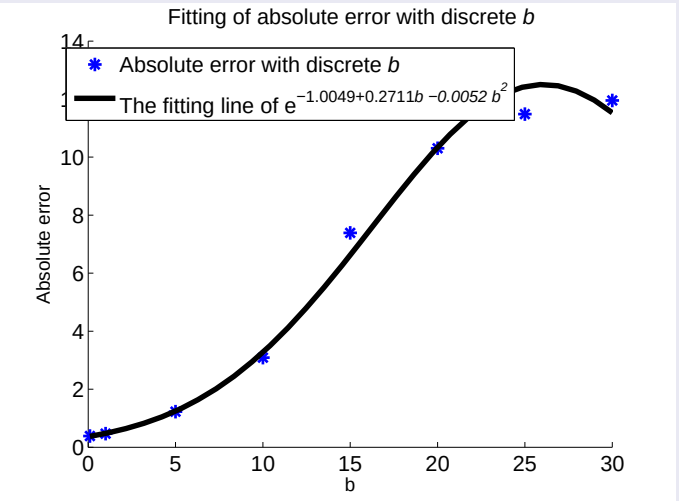


Table: Left: Choices of b and it relative error for exact data. Right: Fitting function $e^{c_1+c_2b+c_3b^2}$ with $c_1 = -1.0049$, $c_2 = 0.2711$ and $c_3 = -0.0052$ for the absolute error.

Outline

1. Introduction
   - Related work
   - Non-radiating source
   - Helmholtz equation
   - Main result
2. Increasing stability with attenuation
   - Analytic functions
   - Exact observability bounds
   - Proof of the main theorem
3. Numerical algorithm for source identification
   - Regularization method
   - Numerical examples
4. Ongoing works

Ongoing work

Ongoing work: Linearized inverse Schrödinger potential problem

Recover the potential function $c(x)$ from Dirichlet-to-Neumann map

$$\begin{cases} -\Delta u - (k^2 - c)u = 0 & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g_0 & \text{on } \partial\Omega, \end{cases}$$

- Bounded exponential functions;
- Infinite measurement at a fixed (large) wavenumber (DtN map);
- Numerical reconstruction algorithm based on the linearization.

Isakov (11); Isakov, Nagayasu, Uhlmann & Wang (14); Isakov & Wang (14); Isakov, Lai & Wang (16); Isakov, Lu & Xu (19, submitted)

Ongoing work

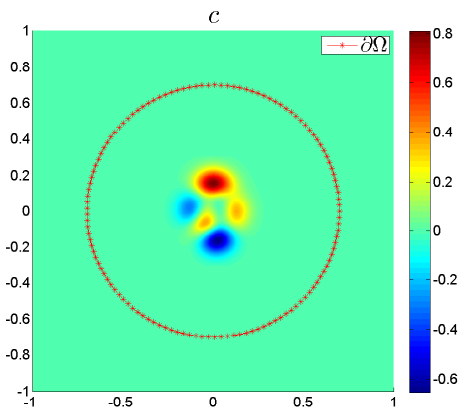
Ongoing work: Linearized inverse Schrödinger potential problem

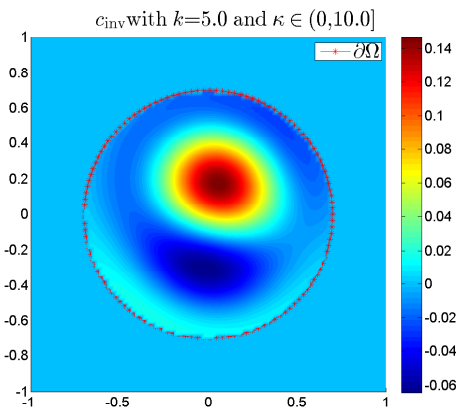
Linearized inverse Schrödinger potential problem:

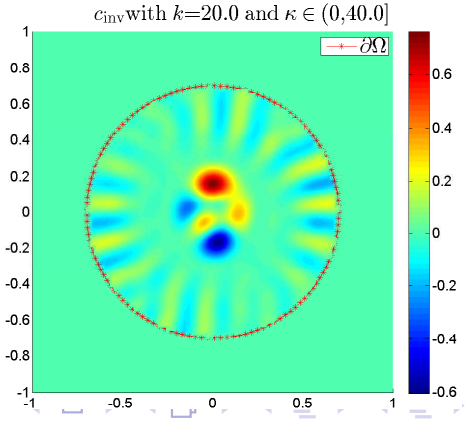
$$(I_0) \quad \begin{cases} -\Delta u_0 - k^2 u_0 = 0 & \text{in } \Omega, \\ u_0 = g_0 & \text{on } \partial\Omega, \end{cases} \tag{1a}$$

$$(I_1) \quad \begin{cases} -\Delta u_1 - k^2 u_1 = -c u_0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1b}$$

Hölder type stability!!







Ongoing work

Ongoing: stochastic source

- Parameter identification of stochastic differential equations

$$\Delta u + k^2 u = f(x) + g(x)\dot{W}_x;$$

- Uncertainty quantification & Data assimilation

$$\begin{cases} \begin{bmatrix} u \\ u' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix} + \begin{bmatrix} 0 \\ f(x) \end{bmatrix} + \begin{bmatrix} 0 \\ g(x)\dot{W}_x \end{bmatrix} \\ v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}. \end{cases}$$

- Fluctuation dissipation theorem (linear response)

 Li (11); Bao, Chow, Li and Zhou (14); Iglesias, Law and Stuart (13); Branicki and Majda (14); Iglesias, Lin, Lu and Stuart (17); Ding, Lu and Cheng (18); Niu, Lu and Cheng (19).

Thank you for your attention.

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