Resonance Analysis of EM Metasurface

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Outline

- Motivation of EM metasurface
- Set-up of EM metasurface
- Approximation of scattered fields
- Plasmon resonance
- Generalizations to various regimes

Motivation for EM Metasurface

- An EM metasurface:
  an artificial sheet of material with sub-wavelength thickness



The Lycurgus Cup
(4th century Roman class cage cup) is encoated with gold nanoparticles

- Plasmon resonance of nanoparticles: a phenomenon caused by the interaction between EM field and nanoparticles in a dielectric medium

- When nanoparticles are excited by EM field whose frequency is close to plasmonic resonance, scattering properties of particles are strongly enhanced

References for Plasmonic Resonance

- Ammari, Millien, Ruiz & Zhang, ARMA 17:
  Mathematical analysis of plasmonic nanoparticles
  $(\text{long, mainly single particle, spectral behaviour in } \omega \text{ and } \delta)$

- Ammari, Ruiz, Yu & Zhang, JDE 16:
  Mathematical analysis of plasmonic resonances for nanoparticles
  (time-hamonic Maxwell, single particle)

Spectral asymptotic analysis in distance between particls:

- Bonnetier & Triki, ARMA 13:
  On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2d $(\omega = 0, \text{ static case, } 2 \text{ particles})$

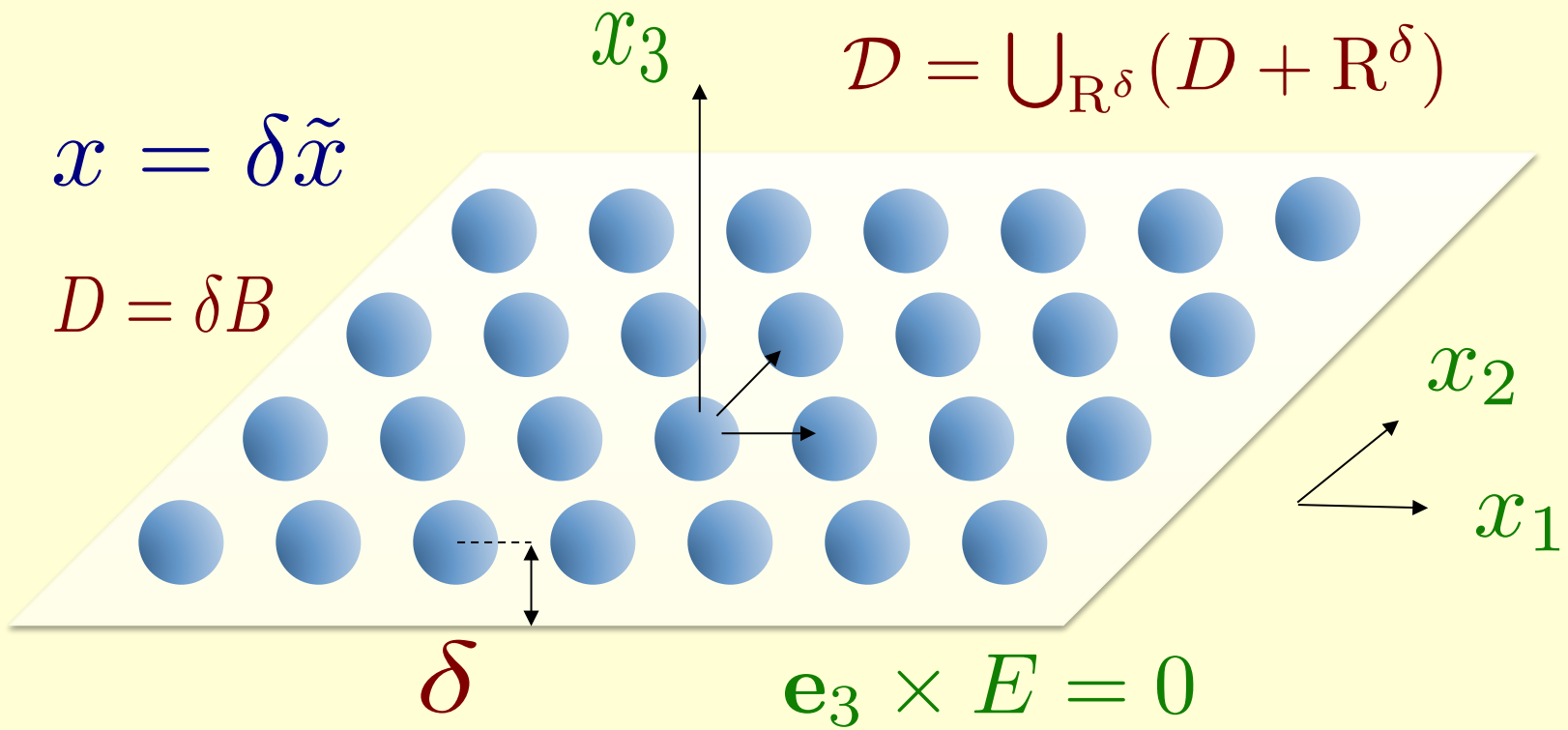
- Bonnetier, Dapogny & Triki, arXiv 17:
  Homogenization of eigenvalues of Neumann-Poincaré operator,
  $(\omega = 0, \text{ static case, periodic distributed particles})$

... ...

Configuration of Nanoparticles

- 2D uniform grid of size δ

$$\Lambda^\delta = \{\mathrm{R}^\delta = n_1\delta\mathbf{a}_1 + n_2\delta\mathbf{a}_2 \in \mathbb{R}^2;\ n_i \in \mathbb{Z}\}$$



Time-harmonic EM System

- Scattering problem:

$$\nabla \times E = ik\mu_{\mathcal{D}} H \quad \text{in} \quad \mathbb{R}^3_+ \backslash \partial\mathcal{D},$$

$$\nabla \times H = -ik\varepsilon_{\mathcal{D}} E \ \text{in} \quad \mathbb{R}^3_+ \backslash \partial\mathcal{D}$$

- Interface conditions $\left(r(u) = \nu \times u \right)$:

$$[r(E)] = [r(H)] = 0 \ \text{ on } \partial\mathcal{D}$$

- Boundary condition:

$$\mathbf{e}_3 \times E = 0 \text{ on } \Gamma$$

with radiation condition

Nanoparticles:
$\varepsilon(\omega)$, $\mu(\omega)$

Quasi-periodic Solutions

- Incident plane wave: $E^i = \mathbf{p} e^{ik\mathbf{d}\cdot\mathbf{x}} - \mathbf{p}^* e^{ik\mathbf{d}^*\cdot\mathbf{x}}$

  ➡ $H^i = \mathbf{d} \times \mathbf{p} e^{ik\mathbf{d}\cdot\mathbf{x}} - \mathbf{d}^* \times \mathbf{p}^* e^{ik\mathbf{d}^*\cdot\mathbf{x}}$

- Find quasi-periodic solution:

  $E(\mathbf{x} + \mathrm{R}^\delta) = e^{i\mathbf{k}\cdot\mathrm{R}^\delta} E(\mathbf{x}), \quad H(\mathbf{x} + \mathrm{R}^\delta) = e^{i\mathbf{k}\cdot\mathrm{R}^\delta} H(\mathbf{x})$

- Rayleigh Bloch expansion of scattered field

  keep all the modes in the expansion:
  decaying exponentially or propagating vertically

  ➡ Radiation condition

- Goal: express E^s in terms of δ, ω,etc.

Analytical Tools

- Useful relations:

$$\Delta_{\partial B} := \nabla_{\partial B} \cdot \nabla_{\partial B} = -\mathrm{curl}_{\partial B} \vec{\mathrm{curl}}_{\partial B}$$

$$\vec{\mathrm{curl}}_{\partial B} \varphi = -r(\nabla_{\partial B} \varphi), \quad \mathrm{curl}_{\partial B} \varphi = -\nabla_{\partial B} \cdot (r\varphi)$$

$$\nabla_{\partial B} \cdot \vec{\mathrm{curl}}_{\partial B} \varphi = 0, \quad \mathrm{curl}_{\partial B} \nabla_{\partial B} \varphi = 0$$

- Useful spaces:

$$H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B) = \{\varphi \in H_T^{-\frac{1}{2}}(\partial B); \quad \nabla_{\partial B} \cdot \varphi \in H^{-\frac{1}{2}}(\partial B)\}$$

$$H_T^{-\frac{1}{2}}(\mathrm{curl}, \partial B) = \{\varphi \in H_T^{-\frac{1}{2}}(\partial B); \quad \mathrm{curl}_{\partial B} \varphi \in H^{-\frac{1}{2}}(\partial B)\}$$

- Helmholtz decomposition:

$$H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B) = \nabla_{\partial B} H_0^{\frac{3}{2}}(\partial B) \oplus \vec{\mathrm{curl}}_{\partial B} H_0^{\frac{1}{2}}(\partial B)$$

Quasi-periodic Green's Tensors

- Quasi-periodic scalar Green's funcs:

$$(\Delta + k^2) G^{\mathbf{k}}_{\#}(\mathbf{x}) = \sum\nolimits_{\mathrm{R}} e^{ik'\cdot x'} \delta_{\mathrm{R}}(\mathbf{x})$$

- Quasi-periodic G funcs with D & N conditions:

$$G^{\mathbf{k}}_{\mathrm{e}}(\mathbf{x},\mathbf{y}) = G^{\mathbf{k}}_{\#}(\mathbf{x}-\mathbf{y}) - G^{\mathbf{k}}_{\#}(\mathbf{x}-\mathbf{y}^*),$$

$$G^{\mathbf{k}}_{\mathrm{m}}(\mathbf{x},\mathbf{y}) = G^{\mathbf{k}}_{\#}(\mathbf{x}-\mathbf{y}) + G^{\mathbf{k}}_{\#}(\mathbf{x}+\mathbf{y}^*)$$

- Electromagnetic quasi-periodic Green's tensors:

$$\mathbb{G}^{\mathbf{k}}_{\mathrm{e/m}}(\mathbf{x},\mathbf{y}) = (1 + k^{-2}\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\cdot)\left[G^{\mathbf{k}}_{\mathrm{e/m}}\mathbf{e}_1,\quad G^{\mathbf{k}}_{\mathrm{e/m}}\mathbf{e}_2,\quad G^{\mathbf{k}}_{\mathrm{m/e}}\mathbf{e}_3\right](\mathbf{x},\mathbf{y})$$

$$\Rightarrow\quad \mathbf{e}_3 \times \mathbb{G}^{\mathbf{k}}_{\mathrm{e}}(\mathbf{x},\mathbf{y}) = 0\,,\quad \mathbf{e}_3 \cdot \mathbb{G}^{\mathbf{k}}_{\mathrm{m}}(\mathbf{x},\mathbf{y}) = 0$$

$$\forall\, \mathbf{x} \in \Gamma,\ \mathbf{y} \in \mathbb{R}^3_+$$

Layer Potential Operators

- Single layer potential $\mathcal{S}^{\mathbf{k}}_{B,\mathrm{e/m}} : H^{-\frac{1}{2}}(\partial B) \longrightarrow H^{\frac{1}{2}}(\partial B)$

$$\mathcal{S}^{\mathbf{k}}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) = \int_{\partial B} G^{\mathbf{k}}_{\mathrm{e/m}}(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})d\sigma$$

- Double layer potential $\mathcal{K}^{\mathbf{k}}_{B,\mathrm{e/m}} : H^{\frac{1}{2}}(\partial B) \longrightarrow H^{\frac{1}{2}}(\partial B)$

$$\mathcal{K}^{\mathbf{k}}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) = \int_{\partial B} \frac{\partial}{\partial \nu_{\mathbf{y}}} G^{\mathbf{k}}_{\mathrm{e/m}}(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})d\sigma$$

- Neumann-Poincaré operator $\mathcal{K}^{\mathbf{k},*}_{B,\mathrm{e/m}} : H^{-\frac{1}{2}}(\partial B) \longrightarrow H^{-\frac{1}{2}}(\partial B)$

$$\mathcal{K}^{\mathbf{k},*}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) = \int_{\partial B} \frac{\partial}{\partial \nu_{\mathbf{x}}} G^{\mathbf{k}}_{\mathrm{e/m}}(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})d\sigma$$

Crucial: spectra ➡ plasmon resonance

Integral Operators with EM Green's Tensors

- $\mathcal{A}^{\mathbf{k}}_{B,\mathrm{e/m}} : H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B) \longrightarrow H(\mathrm{curl}, B)$

$$\mathcal{A}^{\mathbf{k}}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) = \int_{\partial B} \left[G^{\mathbf{k}}_{\mathrm{e/m}} \mathbf{e}_1, G^{\mathbf{k}}_{\mathrm{e/m}} \mathbf{e}_2, G^{\mathbf{k}}_{\mathrm{m/e}} \mathbf{e}_3 \right] (\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d\sigma$$

- $\mathcal{M}^{\mathbf{k}}_{B,\mathrm{e/m}} : H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B) \longrightarrow H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B)$

$$\mathcal{M}^{\mathbf{k}}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) = \int_{\partial B} \nu(\mathbf{x}) \times \nabla_{\mathbf{x}} \times \Pi^{\mathbf{k}}_{\mathrm{e/m}}(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d\sigma$$

- $\mathcal{L}^{\mathbf{k}}_{B,\mathrm{e/m}} : H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B) \longrightarrow H_T^{-\frac{1}{2}}(\mathrm{div}, \partial B)$

$$\mathcal{L}^{\mathbf{k}}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) = \nu(\mathbf{x}) \times (k^2 \mathcal{A}^{\mathbf{k}}_{B,\mathrm{e/m}}[\varphi](\mathbf{x}) + \nabla \mathcal{S}^{\mathbf{k}}_{B,\mathrm{e/m}}[\nabla_{\partial B} \cdot \varphi](\mathbf{x}))$$

Approximation of Scattered Fields

- Write the electric field:

$$E = \begin{cases} E^i + \nabla \times \mathcal{A}^{\mathbf{k}}_{D,\mathrm{m}}[\phi] + \nabla \times \nabla \times \mathcal{A}^{\mathbf{k}}_{D,\mathrm{e}}[\psi], & \mathbf{x} \in \mathbb{R}^3 \backslash \mathcal{D} \\ \mu_c \nabla \times \mathcal{A}^{\mathbf{k}_c}_{D,\mathrm{m}}[\phi] + \nabla \times \nabla \times \mathcal{A}^{\mathbf{k}_c}_{D,\mathrm{e}}[\psi], & \mathbf{x} \in \mathcal{D} \end{cases}$$

solves Maxwell eqns inside & outside particles, & perfect BC

- Using interface conditions

$$(\nu \times \nabla \times \mathcal{A}^{\mathbf{k}}_{B,\mathrm{e/m}})|_{\pm} = \mp \frac{1}{2} + \mathcal{M}^{\mathbf{k}}_{B,\mathrm{e/m}},$$

$$(\nu \times \nabla \times \nabla \times \mathcal{A}^{\mathbf{k}}_{B,\mathrm{e/m}})|_{\pm} = \mathcal{L}^{\mathbf{k}}_{B,\mathrm{e/m}}$$

to get boundary integral eqn:

$$\begin{bmatrix} \frac{\mu_c+1}{2} + \mu_c \mathcal{M}^{\mathbf{k}_c}_{D,\mathrm{m}} - \mathcal{M}^{\mathbf{k}}_{D,\mathrm{m}} & \mathcal{L}^{\mathbf{k}_c}_{D,\mathrm{e}} - \mathcal{L}^{\mathbf{k}}_{D,\mathrm{e}} \\ \mathcal{L}^{\mathbf{k}_c}_{D,\mathrm{m}} - \mathcal{L}^{\mathbf{k}}_{D,\mathrm{m}} & \frac{k^2}{2}(\varepsilon_c + 1) I + k^2 (\varepsilon_c \mathcal{M}^{\mathbf{k}_c}_{D,\mathrm{e}} - \mathcal{M}^{\mathbf{k}}_{D,\mathrm{e}}) \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \nu \times E^i \\ ik\nu \times H^i \end{bmatrix}$$

Leading-order Terms in Densities

- Set $\mathbf{x} = \delta\widetilde{\mathbf{x}}$ to get integral equation on ∂B:

$$\mathcal{W}_{\delta,B}\begin{bmatrix}\widetilde{\phi}\\ \widetilde{\psi}\end{bmatrix} = \begin{bmatrix}\nu(\widetilde{\mathbf{x}}) \times \widetilde{E}^i\\ ik\nu(\widetilde{\mathbf{x}}) \times \widetilde{H}^i\end{bmatrix}$$

- By Taylor expansion of the incident:

$$\begin{bmatrix}\nu(\widetilde{\mathbf{x}}) \times \widetilde{E}^i(\widetilde{\mathbf{x}})\\ ik\nu(\widetilde{\mathbf{x}}) \times \widetilde{H}^i(\widetilde{\mathbf{x}})\end{bmatrix} = \sum_\beta \frac{\delta^{|\beta|}}{\beta!}\begin{bmatrix}\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^\beta \partial^\beta E^i(0)\\ ik\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^\beta \partial^\beta H^i(0)\end{bmatrix}$$

- Then write $\widetilde{\phi} = \sum_\beta \frac{\delta^{|\beta|}}{\beta!}\widetilde{\phi}_\beta, \quad \widetilde{\psi} = \sum_\beta \frac{\delta^{|\beta|}}{\beta!}\widetilde{\psi}_\beta$

hence

$$\mathcal{W}_{\delta,B}\begin{bmatrix}\widetilde{\phi}_\beta\\ \widetilde{\psi}_\beta\end{bmatrix}(\widetilde{\mathbf{x}}) = \begin{bmatrix}\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^\beta \partial^\beta E^i(0)\\ ik\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^\beta \partial^\beta H^i(0)\end{bmatrix}$$

Leading-order Terms in Densities

- Intend to solve

$$\mathcal{W}_{\delta,B}\begin{bmatrix}\widetilde{\phi}_\beta\\ \widetilde{\psi}_\beta\end{bmatrix}(\widetilde{\mathbf{x}})=\begin{bmatrix}\nu(\widetilde{\mathbf{x}})\times\widetilde{\mathbf{x}}^\beta\partial^\beta E^i(0)\\ ik\nu(\widetilde{\mathbf{x}})\times\widetilde{\mathbf{x}}^\beta\partial^\beta H^i(0)\end{bmatrix}$$

- Write

$$\mathcal{W}_{\delta,B}=\sum\nolimits_n\delta^n\mathcal{W}_{n,B}$$

with

$$\mathcal{W}_{0,B}=\begin{bmatrix}\frac{\mu_c+1}{2}+(\mu_c-1)\mathcal{M}_{\mathrm{m}} & (k_c-k)\mathcal{L}_{1,\mathrm{e}}\\ (k_c-k)\mathcal{L}_{1,\mathrm{m}} & k^2\frac{\varepsilon_c+1}{2}I+k^2(\varepsilon_c-1)\mathcal{M}_{\mathrm{e}}\end{bmatrix}$$

$$\mathcal{W}_{n,B}=\begin{bmatrix}(k_c^n-k^n)\mathcal{M}_{n,\mathrm{m}} & (k_c^{n+1}-k^{n+1})\mathcal{L}_{n+1,\mathrm{e}}\\ (k_c^{n+1}-k^{n+1})\mathcal{L}_{n+1,\mathrm{m}} & k^2(\varepsilon_c k_c^n-k^n)\mathcal{M}_{n,\mathrm{e}}\end{bmatrix}$$

- $\mathcal{W}_{0,B}$: diagonal for single particle

Leading-order Terms in Densities

- Recall

$$\mathcal{W}_{\delta,B}\begin{bmatrix}\widetilde{\phi}_\beta\\ \widetilde{\psi}_\beta\end{bmatrix}(\widetilde{\mathbf{x}}) = \begin{bmatrix}\nu(\widetilde{\mathbf{x}})\times\widetilde{\mathbf{x}}^\beta\partial^\beta E^i(0)\\ ik\nu(\widetilde{\mathbf{x}})\times\widetilde{\mathbf{x}}^\beta\partial^\beta H^i(0)\end{bmatrix}$$

- and

$$\mathcal{W}_{\delta,B} = \sum\nolimits_n \delta^n \mathcal{W}_{n,B}$$

- Further expand

$$\widetilde{\phi}_\beta = \sum\nolimits_j \delta^j \widetilde{\phi}_{\beta,j} \quad \& \quad \widetilde{\psi}_\beta = \sum\nolimits_j \delta^j \widetilde{\psi}_{\beta,j}$$

Crucial Parameters

- Two contrast parameters:

$$\lambda_\varepsilon = \frac{1+\varepsilon_c}{2(1-\varepsilon_c)} \quad \& \quad \lambda_\mu = \frac{1+\mu_c}{2(1-\mu_c)}$$

- Two distance parameters:

$$d_\sigma = \min\left\{d(\lambda_\mu, \sigma(\mathcal{K}_{\rm e}^0)), d(\lambda_\varepsilon, \sigma(\mathcal{K}_{\rm m}^0))\right\}$$

$$d_\sigma^* = \min\left\{d(\lambda_\mu, -\sigma(\mathcal{K}_{\rm m}^{0,*})), d(\lambda_\varepsilon, -\sigma(\mathcal{K}_{\rm e}^{0,*}))\right\}$$

Leading-order Terms in Densities

Using expansions of two densities:

$$\widetilde{\phi} = \widetilde{\phi}_{0,0} + \delta(\widetilde{\phi}_{0,1} + \sum_{|\beta|=1} \widetilde{\phi}_{\beta,0}) + O(\delta^2)$$

$$\widetilde{\psi} = \widetilde{\psi}_{0,0} + \delta(\widetilde{\psi}_{0,1} + \sum_{|\beta|=1} \widetilde{\psi}_{\beta,0}) + O(\delta^2)$$

Zero order terms

$$\mathcal{W}_{0,B} \begin{bmatrix} \widetilde{\phi}_{\beta,0} \\ \widetilde{\psi}_{\beta,0} \end{bmatrix} (\widetilde{\mathbf{x}}) = \begin{bmatrix} \nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^{\beta} \partial^{\beta} E^{i}(0) \\ ik\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^{\beta} \partial^{\beta} H^{i}(0) \end{bmatrix}$$

$$\widetilde{\phi}_{\beta,0} = (\lambda_{\mu} - \mathcal{M}_{\mathrm{m}})^{-1} \left(\frac{\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^{\beta} \partial^{\beta} E^{i}(0)}{1-\mu_c} + f_{\beta} \right)$$

$$\widetilde{\psi}_{\beta,0} = (\lambda_{\varepsilon} - \mathcal{M}_{\mathrm{e}})^{-1} \left(\frac{i\nu(\widetilde{\mathbf{x}}) \times \widetilde{\mathbf{x}}^{\beta} \partial^{\beta} H^{i}(0)}{k(1-\varepsilon_c)} + g_{\beta} \right)$$

First-order Terms in Densities

Coupled system:

$$\mathcal{W}_{0,B}\begin{bmatrix}\widetilde{\phi}_{0,1}\\ \widetilde{\psi}_{0,1}\end{bmatrix}+\mathcal{W}_{1,B}\begin{bmatrix}\widetilde{\phi}_{0,0}\\ \widetilde{\psi}_{0,0}\end{bmatrix}=0$$

Componentwise,

$$(\lambda_\mu-\mathcal{M}_{\rm m})[\widetilde{\phi}_{0,1}]+\frac{k_c-k}{1-\mu_c}\mathcal{L}_{1,{\rm e}}[\widetilde{\psi}_{0,1}]+\mathcal{F}(\widetilde{\phi}_{0,0},\widetilde{\psi}_{0,0})=0,$$

$$\frac{k_c-k}{k^2(1-\varepsilon_c)}\mathcal{L}_{1,{\rm m}}[\widetilde{\phi}_{0,1}]+(\lambda_\varepsilon-\mathcal{M}_{\rm e})[\widetilde{\psi}_{0,1}]+\mathcal{G}(\widetilde{\phi}_{0,0},\widetilde{\psi}_{0,0})=0$$

$\Rightarrow$ $\widetilde{\phi}_{0,1}$, $\widetilde{\psi}_{0,1}$ determined once $\widetilde{\phi}_{0,0}$, $\widetilde{\psi}_{0,0}$ available

Will derive explicit representations

Leading-order Terms in Densities

- Using expansions of two densities

$$\widetilde{\phi} = \widetilde{\phi}_{0,0} + \delta(\widetilde{\phi}_{0,1} + \sum_{|\beta|=1} \widetilde{\phi}_{\beta,0}) + O(\delta^2)$$

$$\widetilde{\psi} = \widetilde{\psi}_{0,0} + \delta(\widetilde{\psi}_{0,1} + \sum_{|\beta|=1} \widetilde{\psi}_{\beta,0}) + O(\delta^2)$$

need only surface div for propagative waves:

$$\widetilde{E}^r_p(\widetilde{\mathbf{x}}) = \nabla \times \mathcal{A}^{\delta\mathbf{k}}_{p,\mathrm{m}}[\widetilde{\phi}](\widetilde{\mathbf{x}}) + \delta^{-1}\nabla \times \nabla \times \mathcal{A}^{\delta\mathbf{k}}_{p,\mathrm{e}}[\widetilde{\psi}](\widetilde{\mathbf{x}})$$

- Taking surface div in the coupled system:

$$(\lambda_\mu + \mathcal{K}^{0,*}_{\mathrm{m}})[\nabla_{\partial B} \cdot \widetilde{\phi}_{0,1}] = \frac{k_c^2 - k^2}{1-\mu_c}\gamma_n(\nabla \times \mathcal{A}_{\mathrm{e}}[\widetilde{\psi}_{0,0}]),$$

$$(\lambda_\varepsilon + \mathcal{K}^{0,*}_{\mathrm{e}})[\nabla_{\partial B} \cdot \widetilde{\psi}_{0,1}] = \frac{k_c^2 - k^2}{k^2(1-\varepsilon_c)}\gamma_n(\nabla \times \mathcal{A}_{\mathrm{m}}[\phi_{0,0}])$$

$$\Rightarrow \quad \nabla \times \mathcal{A}^0_{\mathrm{e}}[\widetilde{\psi}_{0,0}] = \nabla u^h\,, \quad \nabla \times \mathcal{A}^0_{\mathrm{m}}[\widetilde{\phi}_{0,0}] = \nabla u^e$$

Plasmon Resonance & Field Enhancement

- Electric and magnetic dipoles:

$$\mathbf{J}_{\mathrm{e}} = \frac{i}{\tau k_3} \int_{\partial B} \widetilde{\mathbf{y}} (\lambda_\varepsilon + \mathcal{K}_{\mathrm{e}}^{0,*})^{-1} [\nu \cdot E^i(0)] d\sigma$$

$$\mathbf{J}_{\mathrm{m}} = -\frac{1}{\tau d_3} \int_{\partial B} \widetilde{\mathbf{y}} (\lambda_\mu + \mathcal{K}_{\mathrm{m}}^{0,*})^{-1} [\nu \cdot H^i(0)] d\sigma$$

- When $\delta/(d_\sigma d_\sigma^*)$ small, and $\mathbf{x}$ away from thin layer:

$$E^r(\mathbf{x}) = \delta k \mathbb{G}_r^{\mathbf{k}}(\mathbf{x}) (i\mathbf{d}^* \times \mathbf{J}'_{\mathrm{m}} + k(\mathbf{J}_{\mathrm{e}})_3 \mathbf{e}_3) + O(\delta^2)$$

- Occurrence of plasmon resonance & field enhancement & order:

$\mathbf{J}_{\mathrm{e}}$ or $\mathbf{J}_{\mathrm{m}}$ blows up when

$$d(\lambda_\varepsilon(\omega), -\mathcal{K}_{\mathrm{e}}^{0,*}) \quad \text{or} \quad d(\lambda_\mu(\omega), -\mathcal{K}_{\mathrm{m}}^{0,*}) \sim 0$$

Generalizations

- Analytical approach applies

  $$\text{particle size} \ll \text{period} \sim \text{wave length}$$

  $$\text{particle size} \ll \text{period} \ll \text{wave length}$$

particles well separated ➡ separated plasmon resonance

- Other boundary conditions on Γ:

  change Green's tensors by the ones with other BCs

- Strongly coupled multi-layers of particles

- Well separated multi-layers of particles

Equivalent Impedance BCs

- On the boundary:

$$\operatorname{curl}_\Gamma u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \forall\, \mathbf{u}$$

$$\vec{\operatorname{curl}}_\Gamma v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}, 0\right) \quad \forall\, v$$

- Set

$$\beta_{\mathrm{e}} = -\frac{1}{\tau}\int_{\partial B} y_3 (\lambda_\varepsilon + \mathcal{K}_{\mathrm{e}}^{0,*})^{-1}[v_3]\, d\sigma$$

$$D_{\mathrm{m}} = \frac{1}{\tau}\int_{\partial B} y' (\lambda_\mu + \mathcal{K}_{\mathrm{m}}^{0,*})^{-1}[\nu']\, d\sigma$$

- non-magnetic plasmonic particles ($\mu_c = 1$):

$$\mathbf{e}_3 \times E^\delta = \delta \frac{i\beta_{\mathrm{e}}}{k} \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma (H^\delta)' \quad \text{on} \quad \Gamma$$

- Magnetic plasmonic particles ($\mu_c \neq 1$):

$$\mathbf{e}_3 \times E^\delta = \delta \frac{i\beta_{\mathrm{e}}}{k} \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma (H^\delta)' - ik\delta D_{\mathrm{m}} (H^\delta)' \quad \text{on} \quad \Gamma$$

Spectra of Neumann-Poincare Operators

- Single layer potential $\mathcal{S}^0_{\rm e/m} : H^{-\frac{1}{2}}(\partial B) \to H^{\frac{1}{2}}(\partial B)$, self-adjoint, &

  Calderón identity $\mathcal{S}^0_{\rm e/m}\mathcal{K}^{0,*}_{\rm e/m} = \mathcal{K}^0_{\rm e/m}\mathcal{S}^0_{\rm e/m}$

- Standard symmetrization via Calderon identity: no work

  $\mathcal{S}^0_{\rm m}$: non-invertible, non-injective, $H^{-\frac{1}{2}}(\partial B)$

- New symmetrization (Ammari-Kang):

$$\widetilde{\mathcal{S}}^0_{\rm e/m}[\psi] = \mathcal{S}^0_{\rm e/m}[\psi] \quad \text{for} \quad \langle \psi, 1\rangle = 0;$$

$$\widetilde{\mathcal{S}}^0_{\rm e/m}[\psi] = 1 \text{ for eigenfunc of } \mathcal{K}^{0,*}_{\rm e/m} \text{ with value } 1/2$$

$$\widetilde{\mathcal{S}}^0_{\rm e/m} : H^{-\frac{1}{2}}(\partial B) \to H^{\frac{1}{2}}(\partial B), \text{ bijective;}$$

$$\Rightarrow \quad \text{Calderón Identity } \widetilde{\mathcal{S}}^0_{\rm e/m}\mathcal{K}^{0,*}_{\rm e/m} = \mathcal{K}^0_{\rm e/m}\widetilde{\mathcal{S}}^0_{\rm e/m}$$

Spectra of Neumann-Poincare Operators

- $\mathcal{H}^*_{\rm e/m}$: space $H^{-\frac{1}{2}}(\partial B)$ with inner products

$$(\phi,\psi)_{\mathcal{H}^*_{\rm e/m}} = -\langle \phi, \widetilde{\mathcal{S}}^0_{\rm e/m}[\psi]\rangle$$

- Then we can symmetrize $\mathcal{K}^{0,*}_{\rm e/m}$:

$\mathcal{K}^{0,*}_{\rm e/m}$: compact, self-adjoint on $\mathcal{H}^*_{\rm e/m}$;

$$\lambda_j^{\rm e/m} \in (-\tfrac{1}{2},\tfrac{1}{2}] : \lambda_0^{\rm e/m} = \tfrac{1}{2},\ \ \lambda_j^{\rm e/m} \to 0$$

$\{\varphi_j^{\rm e/m}\}$: an orthogonal basis in $\mathcal{H}^*_{\rm e/m}$

operators $\mathcal{K}^*_{\rm m}$ & $\mathcal{K}^{0,*}_{\rm m}$ have same spectra

For $\lambda_j \in \sigma(\mathcal{K}^*_{\rm m})\backslash\{0\}$: $\dim ker(\lambda_j - \mathcal{K}^*_{\rm m}) = \dim ker(\lambda_j - \mathcal{K}^{0,*}_{\rm m})$

THANK YOU!