Singularly Perturbed Elliptic Systems Modeling Partial Separation and Their Free Boundaries

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We investigate the asymptotic behavior, as $\beta \to +\infty$, of solutions to competition-diffusion system of type

$$\begin{cases} \Delta u_{i,\beta} = \beta u_{i,\beta} \prod_{j\neq i} u_{j,\beta}^2 & \text{in } \Omega, \\ u_{i,\beta} = \varphi_i \geq 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, 2, 3,$$

where $\varphi_i \in W^{1,\infty}(\Omega)$ satisfy the *partial segregation condition*

$$\varphi_1 \varphi_2 \varphi_3 \equiv 0 \text{ in } \overline{\Omega}.$$

For $\beta > 1$ fixed, a solutions can be obtained as a minimizer of the functional

$$J_\beta(\mathbf{u}, \Omega) := \int_\Omega \big(\sum_{i=1}^3 |\nabla u_i|^2 + \beta \prod_{j=1}^3 u_j^2 \big)\, dx$$

on the set of functions in $H^1(\Omega, \mathbf{R^3})$ with fixed traces on $\partial\Omega$. We prove a *priori* and *uniform* in β Hölder bounds. In the limit, we are lead to minimize the energy

$$J\mathbf{u}, \Omega) := \int_\Omega \sum_{i=1}^3 |\nabla u_i|^2\, dx$$

over all partially segregated states:

$$u_1 u_2 u_3 \equiv 0 \text{ in } \overline{\Omega}$$

satisfying the given, partially segregated, boundary conditions above. We prove regularity of the free boundary up to a low-dimensional singular set.